J. Fluid Mech. (2002), *vol.* 454, *pp.* 287–325. © 2002 Cambridge University Press
DOI: 10.1017/S002211200100708X Printed in the United Kingdom

A Vlasov equation for pressure wave propagation in bubbly fluids

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(Received 9 February 2001 and in revised form 29 August 2001)

The derivation of effective equations for pressure wave propagation in a bubbly fluid at very low void fractions is examined. A Vlasov-type equation is derived for the probability distribution of the bubbles in phase space instead of computing effective equations in terms of averaged quantities. This provides a more general description of the bubble mixture and contains previously derived effective equations as a special case. This Vlasov equation allows for the possibility that locally bubbles may oscillate with different phases or amplitudes or may have different sizes. The linearization of this equation recovers the dispersion relation derived by Carstensen & Foldy. The initial value problem is examined for both ideal bubbly flows and situations where the bubble dynamics have damping mechanisms. In the ideal case, it is found that the pressure waves will damp to zero whereas the bubbles continue to oscillate but with the oscillations becoming incoherent. This damping mechanism is similar to Landau damping in plasmas. Nonlinear effects are considered by using the Hamiltonian structure. It is proven that there is a damping mechanism due to the nonlinearity of single-bubble motion. The Vlasov equation is modified to include effects of liquid viscosity and heat transfer. It is shown that the pressure waves have two damping mechanisms, one from the effects of size distribution and the other from single-bubble damping effects. Consequently, the pressure waves can damp faster than bubble oscillations.

1. Introduction

Air bubbles in bodies of water have dramatic effects on the acoustic properties. For example, the speed of sound in pure water is approximately $1500\,\mathrm{m\,s^{-1}}$, however, in the presence of 0.5% by volume of air bubbles, the sound speed drops to $160\,\mathrm{m\,s^{-1}}$ (at very low frequencies). Bubbles produced by breaking waves are an important source of underwater sound. There has been considerable interest in developing effective equations to model both the propagation of sound in bubbly fluids and the production of sound by bubbles in liquids.

A dispersion relation for sound waves propagating through a bubbly fluid was derived in Carstensen & Foldy (1947) using a linear scattering theory developed by Foldy (1945). Carstensen & Foldy (1947) also presented experimental results of sound scattering from a bubble screen in a lake. Silberman (1957) performed experiments on bubbly fluids in pipes. He measured sound speeds and attenuation and compared his results with a theory developed by Spitzer. Spitzer's results are based on a linear theory and are similar to those presented in Carstensen & Foldy (1947). More recently Cheyne, Stebbings & Roy (1995) measured the sound speed in a bubbly fluid in a pipe using optical techniques.

Iordanskii (1960) and van Wijngaarden (1968) derived a set of effective equations for a nonlinear theory; van Wijngaarden showed, among other things, that traveling wave solutions of these equations can be modelled by the Kortweg–de Vries equation. d'Agostino & Brennen (1989) developed a similar model to study the dynamics of spherical bubble clouds. Noordzij & van Wijngaarden (1974) investigated the effects of bubble movement relative to the liquid and examined the profile of pressure 'shocks' in a bubbly fluid. This problem has been also considered by Gubaidullin, Ivandav, & Nigmatulin (1976) and Nigmatulin (1982); they found that heat transfer between the bubbles and the liquid was very important. This problem was recently considered by Watanabe & Prosperetti (1994) and they also verified the importance of heat transfer. It should be pointed out that in the above papers it is assumed that the bubbles oscillate in a stable fashion. By this we mean that pressures experienced by the bubbles do not cause them to collapse violently or undergo large expansions. The latter situation is associated with the Blake threshold, see for example Prosperetti (1984). The collective collapse of bubbles has been studied by Kuttruff (1999).

Caflisch *et al.* (1985*a*) provided an alternative derivation of the effective equations derived by van Wijngaarden (1968) which among other things elucidated under what scaling conditions these equations could be considered as a good approximation. Caflisch (1985) proved a global existence theory, thus establishing that the effective equations did not form shocks. Miksis & Ting (1986) used homogenization theory to develop effective equations. The dynamics of periodically forced bubble clouds were studied by Smereka & Banerjee (1988) and Birnir & Smereka (1990). Sangani (1991) analytically computed the dispersion relation for bubbly flows considering pair-wise interactions. Sangani & Sureshkumar (1993) also examined sound wave propagation numerically. Zhang & Prosperetti (1994*a*) derived effective equations using the ensemble averaging method developed in Zhang & Prosperetti (1994*b*). They considered bubbles with identical equilibrium radii and derived an extension of the Rayleigh–Plesset equation for the case of interacting bubbles. In addition, they recovered some of Sangani's results (Sangani 1991) when they consider a linearization of their effective equations.

The linear theory developed by Carstensen & Foldy (1947) allowed for the possibility of a size distribution of bubbles, whereas the nonlinear effective equations of Iordanskii (1960) considered a discrete size distribution. Kedrinskii (1968) linearized Iordanskii's equations and computed the phase speed for a continuous size distribution of bubbles. His result for the phase speed was essentially the same as that obtained by Carstensen & Foldy. Commander & Prosperetti (1989) present a new derivation of the dispersion relation of Carstensen & Foldy (1947) based on the work of van Wijngaarden and Caflisch *et al.* (1985*a*) which allows the inclusion of a continuous bubble size distribution. They also make careful comparisons with experiments and find good agreement in many cases. However, for frequencies near the resonant bubble frequency the agreement is not as good. Gavrilyuk (1992) linearized Iordanskii's equations and studied signal propagation in a bubbly fluid with a continuous size distribution of bubbles. He obtained the same result for the phase speed as Kedrinskii (1968). He also found the group speed and studied the asymptotic behaviour of the pressure as $t \to \infty$. Gavrilyuk & Fil'ko (1990) proved that Iordanskii's equations had travelling wave solutions.

The analysis of Carstensen & Foldy was based on linear scattering theory, which supposes linear bubble dynamics. A different approach was put forth independently by Iordanskii (1960) and van Wijngaarden (1968); they derived effective equations which have the advantage of being able to consider the nonlinear bubble dynamics.

Caflisch *et al.* (1985*a*) provided a more rigorous derivation of these effective equations by making two simplifying assumptions: first, that all bubbles have the same equilibrium size and second, the initial conditions of the bubbles vary smoothly in space. Commander & Prosperetti derived effective equations for linear bubble oscillations allowing for a size distribution of bubbles; a derivation based, in part, on the rigorous results of Caflisch *et al.* (1985*a*).

The purpose of this paper is to derive effective equations for a bubbly fluid where the bubbles have an equilibrium size distribution and the possibility of incoherent and nonlinear bubble oscillations. It should be noted that incoherent bubble oscillations can arise through three mechanisms: first, if there is a size distribution of bubbles then nearby bubbles can oscillate at different frequencies; second, if nearby bubbles have different initial conditions different phases and amplitudes can result; lastly, because of nonlinear effects, bubble oscillations of different amplitudes can (even with the same equilibrium size) oscillate with different frequencies. The physical setting we have in mind is the sound production by bubbles in a breaking wave, see, for example Deane (1997) or Medwin & Clay (1998). In this situation, a breaking wave produces a large number of bubbles of different equilibrium sizes, with different phases and amplitudes.

We shall deduce our effective equations using kinetic theory and derive a Vlasov-type equation for the probability distribution of bubbles in phase space. Averaged quantities can then be computed by taking moments. Our model is similar, in spirit, to the Vlasov–Poisson equation of plasma physics and the kinetic equation derived by Russo & Smereka (1996) and Herrero, Lucquin-Desreux & Perthame (1999) to model void wave propagation in incompressible bubbly fluids. The derivation of the effective equations presented by Caflisch *et al.* (1985*a*) was based on Foldy's method in a nonlinear setting. Our derivation is similar to that of Caflisch *et al.* (1985*a*); however we do not assume that bubbles are moving coherently and allow for the possibility that the bubbles have a size distribution. Our new model contains the previous models as a special case. We also show that the appropriate linearization of our model gives the same dispersion relation as presented by Carstensen & Foldy (1947).

We prove that for even ideal bubble oscillations, sound waves will be damped. This damping is shown to be the result of incoherent bubble oscillations. This type of damping is connected with Landau damping of plasmas and the relaxation of coupled oscillators. This conclusion is reached following Smereka (1998) in which a globally coupled Hamiltonian system was studied. In this problem each bubble is a Hamiltonian system and they are coupled to each other through the pressure field. The results in Smereka (1998) are closely related to work found in plasma physics by Landau (1946), Weitzner (1963), Weitzner & Dobrott (1968) and Crawford & Hislop (1989). It is also very closely related to the work of Strogatz, Mathews & Mirollo (1992) on Kuramoto's model.

Let us briefly outline the main new results contained in this work:

A Vlasov equation is derived to model pressure wave propagation in bubbly flows. This model is fully nonlinear and can allow for both a size distribution of bubbles and incoherent bubble oscillations. It is a natural generalization of the equations derived by Iordanskii (1960) and van Wijngaarden (1968) and is a nonlinear version of the equations derived by Commander & Prosperetti (1989).

When linearized this model gives the well-known dispersion relation derived by Carstensen & Foldy (1947).

The initial value problem is studied for the linear model; it is found that in the case when there are no single bubble damping mechanisms, the Fourier transform of the pressure waves will decay by a mechanism similar to Landau damping. As the pressures waves damp the bubble oscillations become incoherent.

The initial value problem is studied for the fully nonlinear problem using the Hamiltonian structure of the equations of motion. It is shown that there is a damping effect caused by the nonlinearity of single bubble motion.

Damping effects are included and it is found that pressure waves will now damp due to single bubble damping and Landau damping.

2. Preliminary remarks

The following equations are in the form derived by Caflisch *et al.* (1985*a*) and are a slight simplification of those derived by van Wijngaarden (1968):

$$\left.\begin{aligned} \frac{1}{\rho_\ell C_\ell^2}\mathscr{P}_t + \nabla\cdot\boldsymbol{u} &= \tfrac{4}{3}\pi n(R^3)_t, \\ \rho_\ell \boldsymbol{u}_t + \nabla\mathscr{P} &= 0, \\ RR_{tt} + \tfrac{3}{2}R_t^2 &= \frac{1}{\rho_\ell}\left[\left(\frac{R_0}{R}\right)^{3\gamma}\mathscr{P}_\infty - \mathscr{P}\right]. \end{aligned}\right\} \tag{2.1}$$

where ρ_ℓ is the density of the liquid, C_ℓ is the speed of sound in the pure liquid, $\mathscr{P}(\boldsymbol{x},t)$ is the average liquid pressure, $\boldsymbol{u}(\boldsymbol{x},t)$ is the average liquid velocity, $n(\boldsymbol{x})$ is the number of bubbles per unit volume, $R(\boldsymbol{x},t)$ is the bubble radius, γ is the adiabatic exponent, R_0 is the equilibrium bubble size and $\mathscr{P}_\infty$ is the ambient pressure. We define the dimensionless number $\chi = \frac{4}{3}\pi n_0\lambda^2 R_0$ where λ is a characteristic length scale and n_0 is the spatial average of n given by

$$n_0 = \frac{1}{V}\int_{\mathscr{V}} n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

where $\mathscr{V}$ is the region of the fluid occupied by the bubbles and its volume is V. One important conclusion of Caflisch *et al.* (1985*a*) is that these equations will be valid if χ is $O(1)$ and $\lambda \gg R_0$. These two conditions imply that the volume fraction of bubbles is very small.

The term $\rho_\ell^{-1}C_\ell^{-2}\mathscr{P}_t$ models the compressibility of the liquid phase. It was shown by Smereka & Banerjee (1988) that it can be a good approximation to ignore the compressibility effects in the liquid provided

$$m \equiv \frac{C_g}{C_\ell}\sqrt{\frac{\rho_g}{\alpha\rho_\ell}} \ll 1,$$

where C_g is the sound speed in the gas, ρ_g is the gas density, and α is the volume fraction of gas. Furthermore, they showed that one must consider time scales larger than m/ω_0^*, where ω_0^* is the natural frequency of a bubble

$$\omega_0^* = \frac{1}{R_0}\sqrt{\frac{3\gamma\mathscr{P}_\infty}{\rho_\ell}}.$$

For air bubbles in water with a void fraction of 1%, $m \approx 0.06$. If the bubble radius is $O(0.1)$ cm then ω_0^* is $O(10^4)\,\mathrm{s}^{-1}$. Hence we are restricted to time scales greater than 10^{-5} s. Consequently under these circumstances it is a good approximation to ignore the term $\rho_\ell^{-1} C_\ell^{-2} \mathscr{P}_t$ in (2.1). By doing this and eliminating $\boldsymbol{u}$ we can rewrite (2.1) as

$$\left.\begin{aligned} -\Delta P &= \tfrac{4}{3}\pi n (R^3)_{tt}, \\ RR_{tt} + \tfrac{3}{2}R_t^2 &= \frac{1}{\rho_\ell}\left[\left(\frac{R_0}{R}\right)^{3\gamma}\mathscr{P}_\infty - \mathscr{P}\right], \end{aligned}\right\} \tag{2.2}$$

where Δ denotes the Laplacian in R^3.

2.1. *Dimensionless form*

Let λ denote the wavelength of a typical disturbance in the system. The dimensionless coordinate, x', is then defined by $x = \lambda x'$. The bubbles are contained in $\mathscr{V}$ which has dimensionless volume $V^* = V/\lambda^3$. Let $\eta(\boldsymbol{x})$ denote the dimensionless bubble density defined by $n = n_0\eta$. It follows η is normalized so that

$$\int \eta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = V^*.$$

Evidently, if the bubbles are uniformly distributed in $\mathscr{V}$, one has

$$\eta(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \mathscr{V} \\ 0, & \text{otherwise.} \end{cases}$$

We introduce dimensionless pressure, bubble radius, and time using $\mathscr{P} = \mathscr{P}_\infty(1+\phi)$, $R = R_0R'$, and $t = (\rho_\ell R_0^2/\mathscr{P}_\infty)^{1/2}t'$ respectively. In these variables (2.2) becomes

$$-\Delta\phi = \chi\eta(R^3)_{tt}, \tag{2.3}$$

$$RR_{tt} + \tfrac{3}{2}R_t^2 = R^{-3\gamma} - 1 - \phi, \tag{2.4}$$

where we have dropped the primes and the reader is reminded that $\chi = \frac{4}{3}\pi n_0\lambda^2 R_0$. The boundary condition for (2.3) is $\phi \to 0$ as $|\boldsymbol{x}| \to \infty$. We note that (2.3) can be written as

$$-\Delta\phi = \chi\eta(3R^2R_{tt} + 6RR_t^2). \tag{2.5}$$

By substituting the expression for R_{tt} from (2.4) into (2.5), we obtain

$$-\Delta\phi + 3\chi\eta R\phi = (\tfrac{1}{2}R_t^2 + R^{-3\gamma} - 1)3\chi\eta R. \tag{2.6}$$

We observe that ϕ represents the perturbation of the pressure about the ambient pressure; therefore ϕ represents sound waves. The equations of motion given by (2.4) and (2.6) can be found in Nigmatulin (1982).

2.1.1. *Linear dispersion relationship*

Consider a spatially uniform cloud of bubbles of infinite extent; then $V \to \infty$ and $\eta = 1$. The equations of motion above have the equilibrium solution $R(\boldsymbol{x}, t) = 1$. If we consider small perturbations $R = 1 + y$ then $y(\boldsymbol{x}, t)$ satisfies the linear equation

$$y_{tt} + 3\gamma y = 9\chi\gamma(-\Delta + 3\chi)^{-1}y. \tag{2.7}$$

We look for solutions of the form $\mathrm{e}^{\mathrm{i}(kx - \Omega t)}$ for (2.7) and find the following dispersion relation:

$$\Omega^2 = \frac{k^2\omega_0^2}{3\chi + k^2}, \tag{2.8}$$

where $\omega_0 = \sqrt{3\gamma}$ is the linear frequency of oscillation of a single bubble. This formula is well known and may be found in the work of Carstensen & Foldy (1947), van

Wijngaarden (1968, 1972) and Caflisch *et al.* (1985*a*), for example. From (2.8) one can calculate the sound speed for very dilute bubbly flow. The sound speed for bubbly flows at higher volume fraction was calculated by Crespo (1969), Caflisch *et al.* (1985*b*) and Sangani (1991). The results of Crespo and Caflisch *et al.* (1985*b*) are valid at low frequencies, whereas Sangani's result is valid over a wide range of frequencies and includes thermal effects. It should be pointed out that the effects of a bubble size distribution has not been considered in the work of Crespo (1969), Caflisch *et al.* (1985*a,b*), Sangani (1991), or Zhang & Prosperetti (1994*a*).

The linearization of (2.3) shows that

$$\phi \propto \mathrm{e}^{\mathrm{i}(kx-\Omega t)}. \tag{2.9}$$

Therefore we see that sound waves in a bubbly fluid are dispersive. It also follows from (2.3) that the Fourier transform of ϕ has the following behaviour:

$$\hat{\phi} = A_p\,\mathrm{e}^{\mathrm{i}\Omega t} + B_p\,\mathrm{e}^{-\mathrm{i}\Omega t}, \tag{2.10}$$

where A_p and B_p are constants that depend on initial conditions. Equation (2.10) indicates that the Fourier modes of $\hat{\phi}$ will oscillate in time.

3. Equations of motion and the continuum limit

Here we present an alternate derivation of (2.3) and (2.4). We will return to dimensional variables and consider a cloud of bubbles located in a region $\mathscr{V}$ of an unbounded fluid. We shall denote the volume of $\mathscr{V}$ as V.

3.1. *Microscopic Lagrangian*

The liquid velocity v_ℓ is the gradient of a velocity potential ϕ; hence

$$v_\ell = \nabla\phi, \quad \boldsymbol{x} \in \mathscr{V}_\ell,$$

where $\mathscr{V}_\ell$ is the domain occupied by the liquid. The velocity potential satisfies the following elliptic problem:

$$\nabla^2\phi = 0, \quad \boldsymbol{x} \in \mathscr{V}_\ell, \tag{3.1}$$

with boundary conditions on the bubble surfaces being

$$\frac{\partial\phi}{\partial n} = \dot{r}_k, \qquad \boldsymbol{x} \in S_k, \quad k = 1, \ldots, N, \tag{3.2}$$

where $\boldsymbol{n}$ is the outward normal from the bubble surface into the liquid. $\dot{r}_k$ is the radial velocity of the bubble and S_k is the surface of the kth bubble. The condition at infinity is

$$\nabla\phi = 0. \tag{3.3}$$

The total kinetic energy of the system is entirely contained in the liquid since the bubbles have no mass. It is given by

$$K = \tfrac{1}{2}\rho_\ell \int_{\mathscr{V}_\ell} |\nabla\phi|^2\,\mathrm{d}\boldsymbol{x}, \tag{3.4}$$

which can be written as

$$K = -\frac{\rho_\ell}{2}\sum_k \int_{S_k} \phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S. \tag{3.5}$$

Our system will have potential energy due to the compressibility of the gas inside the bubbles. We shall model the gas as ideal; therefore the total potential energy is

$$U = 4\pi\mathscr{P}_\infty \sum_k \left(\frac{r_{k0}^{-3\gamma}}{3\gamma - 3} r_k^{-3\gamma+3} + \tfrac{1}{3} r_k^3 \right),$$

where $\mathscr{P}_\infty$ is the equilibrium pressure, r_{k0} is the equilibrium radius of the kth bubble, and γ is the adiabatic exponent. Since we are considering dilute bubbly mixtures, we assume that the distance between bubbles will be large for most bubbles when compared with their radius. Consequently the solution of (3.1) and (3.2) is given approximately by

$$\phi \approx -\sum_k \frac{\dot{r}_k r_k^2}{|\boldsymbol{x} - \boldsymbol{x}_k|}. \tag{3.6}$$

We substitute (3.6) into (3.5) to obtain

$$K \approx 2\pi\rho_\ell \left[\sum_k r_k^3 \dot{r}_k^2 + \sum_{\substack{j,k \\ j \neq k}} \frac{\dot{r}_j \dot{r}_k r_k^2 r_j^2}{|\boldsymbol{x}_k - \boldsymbol{x}_j|} \right].$$

3.1.1. *Dimensionless form*

Let $r_k = r_k' \langle r_0 \rangle$, $\boldsymbol{x}_k = \boldsymbol{x}_k' \lambda$, and $t = t'(\rho_\ell \langle r_0 \rangle^2 / \mathscr{P}_\infty)^{1/2}$, where $\langle r_0 \rangle$ is the average equilibrium bubble size. Then the dimensionless kinetic and potential energies, after dropping the primes, are

$$K = \sum_k \tfrac{1}{2} r_k^3 \dot{r}_k^2 + \frac{\kappa}{2N} \sum_{\substack{j,k \\ j \neq k}} \frac{3\dot{r}_j \dot{r}_k r_k^2 r_j^2}{4\pi |\boldsymbol{x}_k - \boldsymbol{x}_j|} \quad \text{and} \quad U = \sum_k u(r_k, \beta_k), \tag{3.7}$$

where β_k is the dimensionless equilibrium bubble size,

$$u(r, \beta) = \frac{\beta^{3\gamma} r^{-3\gamma+3}}{3\gamma - 3} + \frac{r^3}{3},$$

and

$$\kappa = \frac{4\pi N R_0}{3\lambda}. \tag{3.8}$$

We also note that the dimensionless volume of $\mathscr{V}$ is $V^* = V/\lambda^3$. This indicates that κ and χ are related by

$$\kappa = \chi V^*. \tag{3.9}$$

3.2. *Microscopic equations of motion*

The equations of motion can be deduced by using Hamilton's principle. The dimensionless Lagrangian is

$$L = K - U,$$

where K and U are given by (3.7). The equations of motion are then given by the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{r}_k} - \frac{\partial L}{\partial r_k} = 0$$

from which we find

$$r_k\ddot{r}_k + \tfrac{3}{2}\dot{r}_k^2 = \left(\frac{\beta_k}{r_k}\right)^{-3\gamma} - 1 - \phi_k, \tag{3.10}$$

where

$$\phi_k = \frac{3\kappa}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\neq k}\frac{r_j^2\dot{r}_j}{4\pi|\boldsymbol{x}_k - \boldsymbol{x}_j|}. \tag{3.11}$$

3.3. *Continuum limit*

In this section we shall derive the results of Caflisch *et al.* (1985*a*) in slightly different way. In that work the authors look for a continuum field, $R(\boldsymbol{x}, t)$ such that $r_k(t) = R(\boldsymbol{x}_k, t)$ in the sense that

$$\frac{1}{N}\sum_k (r_k(t) - R(\boldsymbol{x}_k, t))^2 \to 0 \quad \text{as} \quad N \to \infty. \tag{3.12}$$

This is sometimes called a continuum limit. Physically, this implies that bubbles in close proximity to each other have the same equilibrium size and are essentially moving in an identical fashion. By close proximity we refer to bubbles whose distance apart is much less than λ.

It is clear that if we have a distribution of equilibrium bubble sizes which is spatially uniform then it is not possible for (3.12) to be true. Thus in order to take the continuum limit we must invoke the assumption that the bubbles all have the same equilibrium size (the assumption also used by Caflisch *et al.* 1985*a,b*, Sangani 1991 and Zhang & Prosperetti 1994*a*).

Consequently we take $\beta_k = 1$ for all k. It follows from (3.12) that we can write (3.11) as

$$\phi_k = \chi\frac{V^*}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\neq k}\frac{3R^2(\boldsymbol{x}_j, t)R_t(\boldsymbol{x}_j, t)}{4\pi|\boldsymbol{x}_k - \boldsymbol{x}_j|},$$

where (3.9) has been used. Thus it follows (using the same assumptions as Caflisch *et al.* 1985*a*) that

$$\lim_{N\to\infty}\phi_k = \phi(\boldsymbol{x}_k, t) = -\chi\Delta^{-1}\left[\eta(R^3)_{tt}\right]$$

and

$$RR_{tt} + \tfrac{3}{2}R_t^2 = (R^{-3\gamma} - 1 - \phi).$$

These are the same equations, ignoring liquid compressibility, as obtained by Caflisch *et al.* (1985*a*).

The assumption that bubbles have the same equilibrium size is not enough to guarantee that (3.12) will be true; we must also assume that the initial values of r_k and $\dot{r}_k$ satisfy

$$\frac{1}{N}\sum_k \left([r_k(0) - R(\boldsymbol{x}_k, 0)]^2 + [\dot{r}_k(0) - R_t(\boldsymbol{x}_k, 0)]^2\right) \to 0 \quad \text{as} \quad N \to \infty. \tag{3.13}$$

This indicates that bubbles in close proximity to each other must have very similar initial conditions.

Equation (3.12) may also be satisfied when (3.13) is not true in the following case: consider a periodically forced bubble cloud with bubbles of identical sizes with single-bubble damping mechanisms included. Then for large times we could expect (3.12) to be true for small forcing irrespective of (3.13). This would be difficult to prove because the bubble dynamics are nonlinear. For strong enough forcing it has been shown that bubble motion in periodically forced clouds can be chaotic (see, for example, Lauterborn & Suchla 1984 or Smereka, Birnir & Banerjee 1987). This indicates that even if bubbles have initial conditions which are very close their resulting motion can be quite different; consequently (3.12) would not be satisfied.

4. A Vlasov equation

A crucial aspect of the derivation by Caflisch *et al.* (1985*a*) was the existence of a continuum limit (recall (3.12)). Nevertheless, there might be situations (e.g. oceanic bubbles created by a breaking wave or bubbles rising in a pipe) where bubbles in close proximity are oscillating with a different phase and/or amplitude due to different initial conditions or different equilibrium radii. In these situations

$$\frac{1}{N}\sum_k (r_k(t) - R(\boldsymbol{x}_k, t))^2 \neq 0 \quad \text{as} \quad N \to \infty. \tag{4.1}$$

To develop effective equations to handle both (3.12) and (4.1) we shall use kinetic theory. We first define

$$G(\boldsymbol{x}) = \begin{cases} -\dfrac{1}{4\pi|\boldsymbol{x}|}, & |\boldsymbol{x}| \neq 0 \\ 0, & |\boldsymbol{x}| = 0 \end{cases}$$

and write ϕ_k as

$$\phi_k = -\frac{3\kappa}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_j G(\boldsymbol{x}_j - \boldsymbol{x}_k) r_j^2 \dot{r}_j.$$

The equations of motion for N bubbles are written as

$$\dot{r}_k = w_k \quad \text{and} \quad \dot{w}_k = a_k, \tag{4.2}$$

where

$$a_k = r_k^{-1}\left[-\tfrac{3}{2}w_k^2 + \left(\frac{\beta_k}{r_k}\right)^{3\gamma} - 1 - \phi_k\right].$$

Next we let

$$\rho_N(r, w, \boldsymbol{x}, \beta, t) = \frac{V^*}{N}\sum_{k=1}^{N} \delta(r - r_k(t))\, \delta(w - w_k(t))\, \delta(\boldsymbol{x} - \boldsymbol{x}_k)\, \delta(\beta - \beta_k),$$

where ρ_N is the dimensionless bubble density in the phase-space and δ is the Dirac delta function. Macroscopic quantities can be found by integrating over the r, w, and β variables. For example, the dimensionless bubble density is given by

$$\eta(\boldsymbol{x}) = \int \rho_N(r, w, \boldsymbol{x}, \beta, t)\, \mathrm{d}r\, \mathrm{d}w\, \mathrm{d}\beta \tag{4.3}$$

and the average bubble size, $R(\boldsymbol{x}, t)$, is found from

$$R(\boldsymbol{x}, t)\eta(\boldsymbol{x}) = \int \rho_N(r, w, \boldsymbol{x}, \beta, t) r\, \mathrm{d}r\, \mathrm{d}w\, \mathrm{d}\beta. \tag{4.4}$$

The size distribution of bubbles is given by

$$g(\beta) = \frac{\displaystyle\int \rho_N(r, w, \boldsymbol{x}, \beta, t)\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}\boldsymbol{x}}{\displaystyle\int \eta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}},$$

with the normalization conditions

$$\int g(\beta)\,\mathrm{d}\beta = 1 \quad \text{and} \quad \int \beta g(\beta)\,\mathrm{d}\beta = 1. \tag{4.5}$$

We shall assume that the size distribution of bubbles is spatially invariant; therefore

$$\eta(\boldsymbol{x})g(\beta) = \int \rho_N(r, w, \boldsymbol{x}, \beta, t)\,\mathrm{d}r\,\mathrm{d}w.$$

The term $\phi_k(t)$ can be written in terms of ρ_N which is done by defining

$$\phi_N(\boldsymbol{x}, t) = -3\chi\frac{\partial}{\partial t}\int \rho_N(r, w, \boldsymbol{y}, t)r^2 wG(\boldsymbol{y}-\boldsymbol{x})\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\beta;$$

thus we have

$$\phi_k(t) = \phi_N(\boldsymbol{x}_k, t).$$

By taking the time derivative of ρ_N, we find that it must satisfy the following conservation equation

$$\frac{\partial \rho_N}{\partial t} + \frac{\partial(w\rho_N)}{\partial r} + \frac{\partial(a\rho_N)}{\partial w} = 0, \tag{4.6}$$

where

$$a = r^{-1}\left[-\tfrac{3}{2}w^2 + \left(\frac{\beta}{r}\right)^{3\gamma} - 1 - \phi_N(\boldsymbol{x}, t)\right].$$

Next, we let $N \to \infty$ and define $\lim_{N\to\infty}\rho_N = \rho$. We require that the bubble locations be distributed 'smoothly' so that

$$\eta(\boldsymbol{x}) = \lim_{N\to\infty}\int \rho_N(r, w, \boldsymbol{x}, t)\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}\beta$$

is an ordinary function in the sense of distributions. We note that this is a weaker condition than (3.12) and certainly contains (4.1) as a possibility. This derivation is therefore more general than the continuum description provided in van Wijngaarden (1968) and Caflisch *et al.* (1985*a*). The conservation equation for ρ is then

$$\frac{\partial \rho}{\partial t} + \frac{\partial(w\rho)}{\partial r} + \frac{\partial(a\rho)}{\partial w} = 0, \tag{4.7}$$

where

$$\left.\begin{aligned} a &= r^{-1}\left[-\tfrac{3}{2}w^2 + \left(\frac{\beta}{r}\right)^{3\gamma} - 1 - \phi(\boldsymbol{x}, t)\right], \\ \phi &= -3\chi\Delta^{-1}j(\boldsymbol{x}, t) \quad \text{and} \quad j(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\textstyle\int \rho r^2 w\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}\beta. \end{aligned}\right\} \tag{4.8}$$

Substituting the expression for ρ_t into the expression for j, we find that ϕ satisfies the following elliptic equation:

$$-\Delta\phi + 3\chi\eta(\boldsymbol{x})\,R(\boldsymbol{x}, t)\phi = 3\chi B(\boldsymbol{x}, t), \tag{4.9}$$

where

$$B(\boldsymbol{x},t)=\int\left(\tfrac{1}{2}w^2+\left(\frac{\beta}{r}\right)^{3\gamma}-1\right)r\rho(r,w,\boldsymbol{x},t)\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}\beta.$$

R and η are found from (4.3) and (4.4). The derivation of (4.7) and (4.9) is similar to the formulation of a number of different kinetic equations; see, for example, the derivation of the Vlasov–Poisson equation (e.g. Nicholson 1982 or Neunzert 1984), general Vlasov equations (Braun & Hepp 1977), or kinetic equations for bubbly fluids (Herrero *et al.* 1999).

Let us now look for a solution of (4.7) of the form

$$\rho(r,w,\boldsymbol{x},t)=\eta(\boldsymbol{x})\,\delta(\beta-1)\,\delta(r-R(\boldsymbol{x},t))\,\delta(w-R_t(\boldsymbol{x},t)). \tag{4.10}$$

It follows that (4.10) is a weak solution of (4.7) provided that $R(\boldsymbol{x},t)$ satisfies (2.4) and (2.6) (note that (4.9) will reduce to (2.6)). Therefore we see that our new model contains (2.4) and (2.6) as a special case. For this particular solution, we see that all the bubbles at a particular $\boldsymbol{x}$ are behaving identically. Consequently it follows for this particular ρ that (3.12) is satisfied.

Let us briefly comment on the validity of the limit as $N\to\infty$ used in deriving (4.7) from (4.6). This limit relies strongly on the validity of the scaling assumption introduced by Caflisch *et al.* (1985*a*), which is that $n_0\lambda^2R_0$ must be order one. If this scaling is assumed the mathematical issues which arise by taking the limit $N\to\infty$ can be addressed using the approach outlined by Neunzert (1984) in his work on the Vlasov–Poisson equation. This is very technical and is beyond the scope of this paper.

5. Linear dynamics

In this section we shall compare our new model with the work of Carstensen & Foldy (1947) and Commander & Prosperetti (1989). To compare with these works we must first linearize the discrete equations of motion. We let $r_k=\beta_k+r_k'$ and assume $r_k'\ll 1$ and $w_k\ll 1$ to obtain

$$\dot r_k=w_k\quad\text{and}\quad a_k=-\frac{3\gamma r_k}{\beta_k^2}-\frac{\phi_k}{\beta_k}$$

with

$$\phi_k=-\frac{3\kappa}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_j G(\boldsymbol{x}_j-\boldsymbol{x}_k)\beta_j^2w_j,$$

where we have dropped the prime on r_k. In this section and subsequent sections we shall consider a spatially homogeneous cloud of bubbles of infinite extent; hence we have $V\to\infty$ and $\eta(\boldsymbol{x})=1$. By using the same steps outlined in §4 we obtain

$$\frac{\partial\rho}{\partial t}+\frac{\partial(w\rho)}{\partial r}+\frac{\partial(a\rho)}{\partial w}=0, \tag{5.1}$$

where

$$\left.\begin{aligned}a&=-\omega_0^2(\beta)r-\beta^{-1}\phi(\boldsymbol{x},t),\\ -\Delta\phi+3\chi\phi&=-3\chi B(\boldsymbol{x},t),\\ B(\boldsymbol{x},t)&=-\int\beta^2\omega_0^2(\beta)r\rho\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}\beta,\end{aligned}\right\} \tag{5.2}$$

and

$$\omega_0(\beta) = \frac{\sqrt{3\gamma}}{\beta}. \tag{5.3}$$

Next we consider the quantities

$$\bar{r} = \bar{r}(\boldsymbol{x}, t, \beta) = \frac{\displaystyle\int r\rho \,\mathrm{d}r \,\mathrm{d}w}{\displaystyle\int \rho \,\mathrm{d}r \,\mathrm{d}w} = \frac{1}{g(\beta)} \int r\rho \,\mathrm{d}r \,\mathrm{d}w$$

and

$$\bar{w} = \bar{w}(\boldsymbol{x}, t, \beta) = \frac{1}{g(\beta)} \int w\rho \,\mathrm{d}r \,\mathrm{d}w;$$

$\bar{r}$ and $\bar{w}$ are the phase-averaged bubble radius and radial velocity. By taking moments of (5.1) we find the time evolution of $\bar{r}$ and $\bar{w}$ to be

$$\frac{\partial \bar{r}}{\partial t} - \bar{w} = 0, \quad \frac{\partial \bar{w}}{\partial t} + \omega_0^2(\beta)\bar{r} + \frac{\phi}{\beta} = 0.$$

These equations can be combined to give

$$\frac{\partial^2 \bar{r}}{\partial t^2} + \omega_0^2(\beta)\bar{r} + \frac{\phi}{\beta} = 0. \tag{5.4}$$

The expression for ϕ becomes

$$-\Delta\phi + 3\chi\phi = -3\chi \int \beta^2 \omega_0^2(\beta) g(\beta) \bar{r} \,\mathrm{d}\beta. \tag{5.5}$$

We note that (5.5) can be rewritten using (5.4) and (4.5) as

$$-\Delta\phi = -3\chi \int \beta^2 \frac{\partial^2 \bar{r}}{\partial t^2} g(\beta) \,\mathrm{d}\beta. \tag{5.6}$$

This is the dimensionless form of the pressure equation in the incompressible limit ($C_\ell \to \infty$) obtained by Commander & Prosperetti (1989). If we consider ideal bubble oscillations then (5.4) is the same equation as obtained by Commander & Prosperetti. Therefore we see that we can systematically derive a closed set of equations in terms of phase-averaged quantities.

5.1. *Dispersion relationship*

In this section we shall compute the dispersion relation for (5.4) and (5.5) and examine its properties. We begin by looking for solutions of the form

$$\phi \propto \mathrm{e}^{\mathrm{i}(kx - \Omega t)} \quad \text{and} \quad \bar{r} \propto \mathrm{e}^{\mathrm{i}(kx - \Omega t)}, \tag{5.7}$$

where Ω is real and $k = k_R + \mathrm{i}k_I$ is complex. We remark that since k is complex it follows that

$$\phi \propto \mathrm{e}^{-k_I x} \, \mathrm{e}^{(\mathrm{i}k_R x - \Omega t)}$$

corresponds to a solution that decays in space. If we substitute (5.7) into (5.4) and (5.5) we find solutions of the desired form, provided the following dispersion relation is satisfied:

$$k^2 = 3\chi \int \frac{\Omega^2 \beta g(\beta)}{\omega_0(\beta) - \Omega^2} \,\mathrm{d}\beta.$$

This expression is not well defined and is reminiscent of the dispersion relation calculated by Vlasov for the Vlasov–Poisson equation. This was shown to be incorrect by Landau (1946) and he provided the correct derivation by studying the initial value problem. We shall do a similar calculation in § 5.2 and show that this expression should be

$$k^2 = 3\chi \,⨍ \frac{\Omega^2 \beta g(\beta)}{\omega_0^2(\beta) - \Omega^2}\,\mathrm{d}\beta + \mathrm{i}\frac{\pi\,\mathrm{sgn}(\Omega) 3\chi \Omega^2 \beta_\Omega g(\beta_\Omega)}{2\omega_0(\beta_\Omega)|\omega_0'(\beta_\Omega)|}, \tag{5.8}$$

where ⨍ denotes the principal value integral and β_Ω is the resonant bubble frequency which satisfies

$$\omega_0(\beta_\Omega) = \Omega. \tag{5.9}$$

Using the expression for ω_0, we find

$$\beta_\Omega = \frac{\sqrt{3\gamma}}{\beta}. \tag{5.10}$$

We also determine that (5.8) can be written as

$$k^2 = 3\chi \,⨍ \frac{\Omega^2 \beta g(\beta)}{\omega_0^2(\beta) - \Omega^2}\mathrm{d}\beta + \mathrm{i}\frac{\pi}{2}\,\mathrm{sgn}(\Omega)\, 3\chi \beta_\Omega^2 g(\beta_\Omega) \tag{5.11}$$

$$= A + \mathrm{i}\,\mathrm{sgn}(\Omega) B. \tag{5.12}$$

5.1.1. *Comparison with previous work*

Carstensen & Foldy (1947) present the following dispersion relation for a dilute bubbly fluid:

$$k^2 = k_0^2 + \int \frac{4\pi R n(R)}{(\omega_0^2(R)/\Omega^2) - 1 + \mathrm{i}b}\,\mathrm{d}R, \tag{5.13}$$

where k is the wavenumber, $k_0 = \Omega/C_\ell$, R is the equilibrium bubble radius, $\omega_0(R)$ is the bubble oscillation frequency, Ω is the imposed frequency, and b is the damping coefficient. This equation is in dimensional form and is equivalent to equations (9-5) and (9-6) of Carstensen & Foldy (1947). We will consider the incompressible limit ($C_\ell \to \infty$) and rewrite (5.13) in dimensionless variables to obtain

$$k^2 = 3\chi \int \frac{\beta g(\beta)}{(\omega_0^2(\beta)/\Omega^2) - 1 + \mathrm{i}b}\,\mathrm{d}\beta. \tag{5.14}$$

We shall now examine (5.14) in the limit as $b \to 0$. It is decomposed into its real and imaginary parts to obtain

$$k^2 = (k_R + \mathrm{i}k_I)^2 = A_b - \mathrm{i}B_b, \tag{5.15}$$

where

$$A_b = 3\chi \int \frac{\left[\omega_0^2(\beta)/\Omega^2) - 1\right]\beta g(\beta))}{\left[(\omega_0^2(\beta)/\Omega^2) - 1\right]^2 + b^2}\,\mathrm{d}\beta \quad \text{and} \quad B_b = 3\chi \int \frac{b\beta g(\beta)}{\left[(\omega_0^2(\beta)/\Omega^2) - 1\right]^2 + b^2}\,\mathrm{d}\beta.$$

These are the dimensionless forms of equations (9–107) and (9–111) given by Carstensen & Foldy (1947). Next we let $b \to 0$ and find

$$\lim_{b\to 0} A_b = A_0 = 3\chi \,⨍ \frac{\Omega^2 \beta g(\beta)}{\omega_0^2(\beta) - \Omega^2}\,\mathrm{d}\beta \quad \text{and} \quad \lim_{b\to 0} B_b = B_0 = \tfrac{3}{2}\chi\pi\beta_\Omega^2 g(\beta_\Omega), \tag{5.16}$$

where $\beta_\Omega = \sqrt{3\gamma}/\Omega$ is the resonant bubble size and satisfies the equation $\omega_0(\beta_\Omega) = \Omega$. The expression for B given by (5.16) is the dimensionless form of (9-112) derived by Carstensen & Foldy (1947).

5.1.2. *Example*

We will now examine in more detail the above results in the case when the bubble size distribution is of the form

$$g_\varepsilon(\beta) = \frac{1}{\varepsilon} h\left(\frac{\beta - 1}{\varepsilon}\right), \tag{5.17}$$

where $h(x)$ has a single maximum at $x = 0$. For example, $h(x)$, could be

$$h(x) = \begin{cases} \frac{3}{4}\left(1 - x^2\right), & |x| \leqslant 1 \\ 0, & |x| > 1. \end{cases} \tag{5.18}$$

If we use (5.18) in (5.17) then $g_\varepsilon(\beta)$ represents the situation where the bubbles have a size distribution centred at $\beta = 1$, the range of bubbles sizes is between $1 - \varepsilon$ and $1 + \varepsilon$. The spread of the distribution is controlled by ε and as $\varepsilon \to 0$ all of the bubbles have the same size.

We now explicitly compute the dispersion relation, (5.11), for the bubble size distribution given by using (5.18) in (5.17) with $\varepsilon < 1$. We find

$$k^2 = A\left(\frac{\Omega}{\sqrt{3\gamma}}\right) + \mathrm{i}\,\mathrm{sgn}(\Omega)\, B\left(\frac{\Omega}{\sqrt{3\gamma}}\right) \tag{5.19}$$

with

$$A(\varpi) = -3\chi + \frac{9\chi}{8\varepsilon^3\varpi^4}\left(-4\varepsilon\varpi^2 + c_1 \log\left|\frac{1 - \varpi - \varepsilon\varpi}{1 - \varpi + \varepsilon\varpi}\right| + c_2 \log\left|\frac{1 + \varpi + \varepsilon\varpi}{1 + \varpi - \varepsilon\varpi}\right|\right)$$

and

$$B(\varpi) = \frac{3\pi\chi}{2\varpi^2} g_\varepsilon\left(\frac{1}{\varpi}\right),$$

where $c_1 = (1 - 2\varpi + \varpi^2 - \varepsilon^2\varpi^2)$ and $c_2 = (1 + 2\varpi + \varpi^2 - \varepsilon^2\varpi^2)$. We determine k_R and k_I from (5.19) and these are plotted on figures 1(*a*) and 1(*b*). We remark that these plots will not change in any qualitative way if we use a slightly different functional form for $h(x)$. We recall that the values of k_R and k_I correspond to solution of the form

$$\phi(\boldsymbol{x}, t) \propto \mathrm{e}^{-k_I x} \mathrm{e}^{\mathrm{i}(k_R x - t\Omega)}. \tag{5.20}$$

These solutions correspond to time-periodic solutions of (5.4) and (5.5). For example, suppose we consider a one-dimensional bubble cloud in the region $0 < x < \infty$ with the pressure boundary condition $\phi(0, t) = A\cos(\Omega t)$. The time-periodic solution is

$$\phi(x, t) = A\,\mathrm{e}^{-k_I x} \cos(k_R x - t\Omega),$$

where k_I, k_R, and Ω satisfy (5.11).

We note that if $\varepsilon \to 0$ then $B \to 0$. This is the situation where the bubbles all have the same size. In this case, one can show that

$$k_R = \begin{cases} \sqrt{\dfrac{3\chi\Omega^2}{3\gamma - \Omega^2}}, & \Omega < \sqrt{3\gamma} \\ 0, & \Omega \geqslant \sqrt{3\gamma} \end{cases} \quad \text{and} \quad k_I = \begin{cases} 0, & \Omega \leqslant \sqrt{3\gamma} \\ \sqrt{\dfrac{3\chi\Omega^2}{\Omega^2 - 3\gamma}}, & \Omega > \sqrt{3\gamma}. \end{cases} \tag{5.21}$$

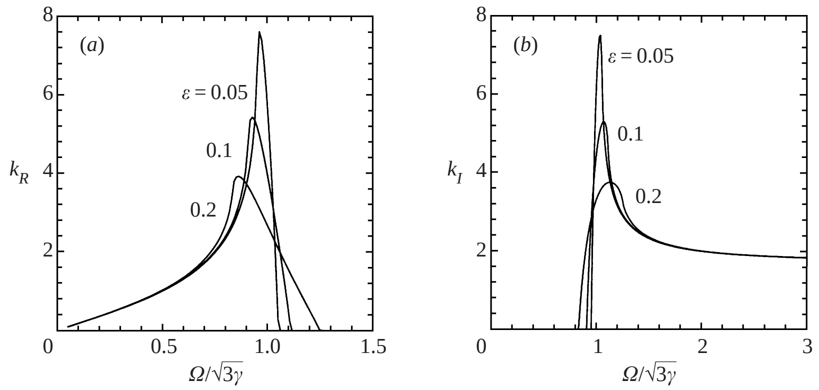

FIGURE 1. (a) The real part and (b) the imaginary part of the wavenumber vs. frequency, plotted for several values of ε. We have chosen $\chi = 1$ and $\varepsilon = 0.05, 0.1$ and 0.2

Thus when all of the bubbles have the same size sound will propagate if the imposed frequency, Ω, is less than the natural frequency of a single bubble, $\sqrt{3\gamma}$. On the other hand, if $\Omega > \sqrt{3\gamma}$ then the pressure field will attenuate exponentially. This is the well-known cut-off phenomenon which has been explored by Smereka & Banerjee (1988) for periodically excited bubble clouds. The effect of a bubble size distribution on this phenomenon is quite interesting. Figure 1 shows that when $\Omega < \sqrt{3\gamma}$ sound waves will propagate but can experience a small spatial decay if the frequency is close to the resonant bubble frequency. On the other hand these results show that when $\Omega > \sqrt{3\gamma}$, the sound will still be attenuated but a small signal can be propagated provided the frequency is close to the resonant bubble frequency.

Finally we emphasize that the spatial attenuation of the pressure waves occurs whether or not there is a size distributions of bubbles. This is clear from both (5.21) and figure 1(b). Let us also stress that this decay should in no way be confused with Landau damping. The spatial attenuation reflects the fact that the bubbly mixture is a dispersive medium that cannot propagate signals above the resonant frequency of the bubbles. An elementary discussion of this cut-off phenomenon can be found in French (1971). In this problem Landau damping is associated with the temporal decay of the Fourier transform of the pressure waves. In the next section we shall prove that Landau damping will be observed if there is a continuous bubble size distribution. Furthermore, Landau damping will not occur if the bubbles all have the same equilibrium size. As pointed out by Landau (1946) this effect is best understood by solving the initial value problem.

5.2. *Initial value problem*

In this section we shall examine the initial value problem for (5.4) and (5.5). The physical situation is as follows: we have an infinite expanse of bubbles with a continuous distribution of bubble sizes with arbitrary initial conditions given in terms of a probability density in phase space. This could be thought of as an idealized cloud of bubbles produced by a breaking wave.

We observe that (5.4) is a simple harmonic oscillator for the phase-averaged bubble radius driven by the sound waves. The sound waves are determined by the collective motion as described by (5.5). It is therefore apparent that there are two basic modes of motion which are individual bubble oscillations and collective motion. The dynamics are then a superposition of these modes; a given bubble will move with two modes of motion, one associated with the collective motion and the other associated with its own natural frequency. The mode of motion where the bubbles respond to the collective motion is a coherent oscillation since nearby bubbles will be moving in unison. The mode of motion where the bubbles move as individual bubble oscillations can result in incoherent motion since nearby bubbles may not be moving together. It is evident from (5.5) that sound waves are associated with the collective modes.

The purpose of §§ 5.2 and 5.3 is to examine how these modes interact. We shall prove that the collective modes will decay to zero when there is a continuous size distribution of bubbles. In particular we shall see that the Fourier modes of the pressure waves decay to zero even though the bubble oscillations persist. The size distribution of the bubbles causes them to oscillate out of phase with each other, ultimately resulting in the bubble oscillations becoming completely incoherent and thus not producing any sound. This damping mechanism is very closely related to Landau damping in plasmas as well as the relaxation process discussed by Smereka (1998).

We begin by taking the Fourier transform in space of (5.4) and (5.5) to obtain

$$\frac{\partial^2 \hat{\hat{r}}}{\partial t^2} + \omega_0^2(\beta)\hat{\hat{r}} + \frac{\hat{\phi}}{\beta} = 0 \tag{5.22}$$

and

$$\hat{\phi} = -\frac{3\chi}{k^2 + 3\chi} \int \beta^2 \omega_0^2(\beta)\hat{\hat{r}} g(\beta)\,\mathrm{d}\beta, \tag{5.23}$$

where $\hat{}$ is defined as $\hat{f} = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x$.

Now we study the initial value problem of (5.22) and (5.23) using Laplace transforms. Using the approach of Smereka (1998, henceforth referred to as Ref. 1) we obtain the following expression:

$$\hat{\phi} = \frac{1}{2\pi\mathrm{i}} \int_{-\infty+\mathrm{i}\Lambda}^{\infty+\mathrm{i}\Lambda} \frac{T(z)}{D(z)}\,\mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}z. \tag{5.24}$$

In the above integral, Λ is sufficiently large so that the path of integration is above any singularities of the integrand. We remark that our Laplace transform variable is z which is the usual Laplace transform variable multiplied by i. The functions in the integrand are defined as

$$T(z) = 3\chi \int_0^{\infty} \frac{\beta^2 \omega_0^2(\beta) g(\beta)}{\omega_0^2(\beta) - z^2}(z\hat{\hat{r}}_0 - \mathrm{i}\widehat{\hat{w}}_0)\,\mathrm{d}\beta \tag{5.25}$$

and

$$D(z) = k^2 - 3\chi \int_0^{\infty} \frac{z^2 \beta g(\beta)}{\omega_0^2(\beta) - z^2}\,\mathrm{d}\beta, \tag{5.26}$$

where $\hat{\hat{r}}_0$ and $\widehat{\hat{w}}_0$ are the initial values of $\hat{\hat{r}}$ and $\widehat{\hat{w}}$.

The goal of this section will be to understand the behaviour of $\hat{\phi}$. It is a simple matter to verify, with the assumptions we have previously made, that both $T(z)$ and $D(z)$ are analytic functions provided $\mathrm{Im}(z) \neq 0$. It is shown in Appendix A that $D(z)$ has no zeros. Thus the integrand of (5.24) is analytic provided $\mathrm{Im}(z) \neq 0$. We let $z = u + \mathrm{i}v$; it is shown in Ref. 1 that we can write $\hat{\phi}$ as

$$\hat{\phi} = \int_{-\infty}^{\infty} J(u)\,\mathrm{e}^{-\mathrm{i}ut}\,\mathrm{d}u, \tag{5.27}$$

where $2\pi\mathrm{i}J(u)$ denotes the jump of T/D across the u-axis. A similar construction was also used by Weitzner (1963), Weitzner & Dobrott (1968) and Crawford & Hislop (1989). It is shown in Ref. 1 that $J(u)$ is an L^1 function provided we make some reasonable assumptions on $g(\beta)$ (see Ref. 1 for details). It therefore follows from the Riemann–Lesbegue lemma that

$$\hat{\phi} \to 0 \quad \text{as} \quad t \to \infty. \tag{5.28}$$

We have therefore proved that the Fourier transform of the sound waves will decay in time to zero. However when bubbles have identical equilibrium sizes the Fourier transform of the pressure wave will not decay (see (2.10)). Therefore the effect of a size distribution is to cause the Fourier transform of the pressure wave to decay. This is somewhat surprising since there are no damping mechanisms for the bubble motion. In fact we will show in the next section that the bubble oscillations do not damp but instead become desynchronized which causes the pressure waves to decay. This decay mechanism is very similar to Landau damping in collisionless plasmas. Landau (1946) showed that the Fourier transform of the electric field will decay exponentially in time despite the fact that the electrons in the plasma have no damping mechanisms.

If we define the total energy of the sound field to be

$$E_s(t) = \int_{-\infty}^{\infty} |\phi|^2 \, \mathrm{d}x$$

then (5.28) combined with Parseval's Theorem implies that the energy of the sound field also decays to zero as $t \to \infty$ when there is a continuous size distribution of bubbles. On the other hand, it follows from (2.10) that $E_s(t)$ will not decay when the bubbles have identical equilibrium sizes.

5.2.1. *Exponential decay*

We have just shown that $\hat{\phi} \to 0$ as $t \to \infty$. We would like to understand the exact nature of the relaxation of $\hat{\phi}$ to zero. We find that $\hat{\phi}$ can decay exponentially fast on an intermediate time scale. To understand this it is useful to compute $J(u)$. It is shown in Ref. 1 that

$$\lim_{v \to 0^+} T(u \pm \mathrm{i}v) = T_1(u) \pm \mathrm{i}T_2(u) \tag{5.29}$$

and

$$\lim_{v \to 0^+} D(u \pm \mathrm{i}v) = D_R(u) \pm \mathrm{i}D_I(u). \tag{5.30}$$

T_1 and T_2 are complex whereas D_R and D_I are both real. It is also shown that

$$D_R(u) = k^2 - 3\chi \fint \frac{u^2 \beta g(\beta)}{\omega_0^2(\beta) - u^2} \, \mathrm{d}\beta \tag{5.31}$$

with

$$D_I(u) = \frac{3\chi\pi \, \mathrm{sgn}(u) u^2 \beta_u g(\beta_u)}{|\omega_0'(\beta_u)| \omega_0(\beta_u)}, \tag{5.32}$$

where β_u is the single positive root of the equation

$$\omega_0(\beta_u) = |u|. \tag{5.33}$$

Using (5.3), we find

$$D_I(u) = -\frac{9\pi\chi\gamma \, \mathrm{sgn}(u)}{2u^2} g\left(\frac{\sqrt{3\gamma}}{|u|}\right). \tag{5.34}$$

We can compute similar expressions for T_1 and T_2 but we will not need them for the present discussion. Using (5.29) and (5.30), we find

$$J(u) = \frac{\tau(u)}{\pi d(u)} \quad \text{where} \quad \tau(u) = D_R T_2 - T_1 D_I \quad \text{and} \quad d(u) = D_R^2 + D_I^2.$$

Recall that $D(z)$ has no zeros; therefore $d(u)$ cannot vanish; it can, however, become small. Therefore $J(u)$ will become large and the dominant contribution to $\hat{\phi}$ will occur in the vicinity of the maximum under certain conditions which we make precise below. Suppose $d(u)$ has a local minimum at u_0. Since $d(u)$ is even then it must have one at $-u_0$. It follows from Ref. 1 that

$$\hat{\phi} = \mathrm{e}^{-\delta t}\left[A(u_0)\,\mathrm{e}^{-\mathrm{i}u_0 t} + A(-u_0)\,\mathrm{e}^{\mathrm{i}u_0 t}\right] \quad t = O(\delta^{-1}) \quad \text{as} \quad \delta \to 0, \tag{5.35}$$

where

$$\delta = \sqrt{\frac{d(u_0)}{a}}, \quad A(u_0) = \frac{\tau(u_0)}{a\sqrt{d(u_0)}} - \mathrm{i}\frac{\tau'(u_0)}{a^2} \quad \text{and} \quad a^2 = \tfrac{1}{2}d''(u_0). \tag{5.36}$$

Thus $\hat{\phi}$ will decay exponentially on an intermediate time scale.

Let us now compute $\hat{\phi}$ in the situation when

$$g(\beta) = g_\varepsilon(\beta),$$

where g_ε is given by (5.17). We can apply techniques similar to those used in Appendix D to show for $\varepsilon \ll 1$ that

$$D_R(u) = k^2 - \frac{9\gamma\chi}{3\gamma - u^2} + O(\varepsilon^2). \tag{5.37}$$

We find that $D_R(\pm\Omega) = 0$ if

$$\Omega = \frac{k\sqrt{3\gamma}}{\sqrt{k^2 + 3\chi}} + O(\varepsilon^2). \tag{5.38}$$

We now use (5.38) and (5.34) to show that

$$|D_I(\Omega)| \approx \frac{3\chi\pi}{2k^2}(k^2 + 3\chi)\, g\left(\frac{\Omega}{\sqrt{3\gamma}}\right).$$

It is easy to verify that

$$|D_I(\Omega)| \ll 1 \quad \text{provided} \quad \varepsilon \ll 1. \tag{5.39}$$

It is apparent from (5.39) and the fact $D_R(\pm\Omega) = 0$, that $d(u) = D_R^2(u) + D_I^2(u)$ has a minimum at $u \approx \pm\Omega$. Furthermore one can show that

$$d''(\Omega) \approx 2(D_R'(\Omega))^2.$$

We use (5.37) and (5.38) to show that

$$|D_R'(\Omega)| = \frac{2k(3\chi + k^2)^{3/2}}{3\chi\sqrt{3\gamma}} + O(\varepsilon^2).$$

Thus we find from (5.36) that

$$\delta \approx \frac{(3\chi)^2\pi}{4k^3}\sqrt{\frac{3\gamma}{k^2 + 3\chi}}\, g_\varepsilon\left(\frac{\sqrt{k^2 + 3\chi}}{k}\right). \tag{5.40}$$

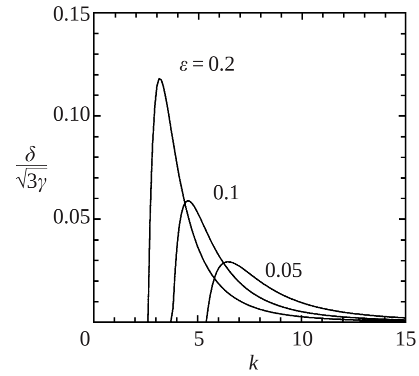

FIGURE 2. The damping rate vs. wavenumber. We have chosen $\chi = 1$ and $\varepsilon = 0.05, 0.1$ and 0.2.

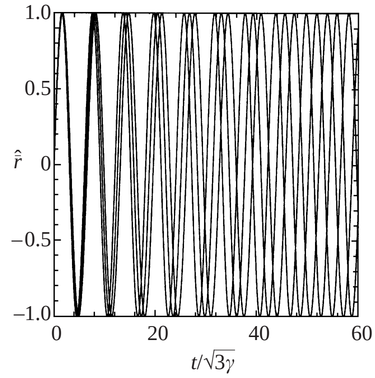

FIGURE 3. $\hat{r}(\beta, t) = \sin(\omega_0(\beta)t)$ is plotted as a function of time for $\beta = 0.96, 0.98, 1.00, 1.02$ and 1.04.

One can prove for g_ε that

$$\lim_{\varepsilon \to 0} g_\varepsilon \left(\frac{\sqrt{k^2 + 3\chi}}{k} \right) = 0;$$

hence it follows that $\delta \ll 1$ if $\varepsilon \ll 1$. Therefore for $\varepsilon \ll 1$, it follows from (5.35) and (5.36) that

$$\hat{\phi} = \mathrm{e}^{-\delta t} \left(A_p \, \mathrm{e}^{\mathrm{i}\Omega t} + B_p \, \mathrm{e}^{-\mathrm{i}\Omega t} \right) \quad \text{if} \quad t = O(\delta^{-1}), \tag{5.41}$$

where Ω and δ are given above. A_p and B_p are constants that depend on the initial conditions and the bubble size distribution. Therefore we see that the Fourier transform of the pressure wave will damp exponentially when the bubbly fluid contains a continuous distribution of bubble sizes. The decay rate depends on both k and the bubble size distribution. We have plotted δ versus k in figure 2 for various values of ε.

5.3. *Bubble behaviour*

In this section we shall examine the bubble motion following a similar procedure as outlined in the previous section. Using Laplace transforms (see Ref. 1), we find that

$$\hat{r} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\Lambda}^{\infty + \mathrm{i}\Lambda} R(z) \, \mathrm{e}^{-\mathrm{i}zt} \, \mathrm{d}z \tag{5.42}$$

where

$$R(z) = \frac{1}{z^2 - \omega_0^2(\beta)} \left(-\mathrm{i}z\hat{\hat{r}}_0 + \hat{\hat{w}}_0 + \frac{3\chi}{\beta} \frac{T(z)}{D(z)} \right).$$

The important difference between (5.42) and (5.24) is that (5.42) has simple poles at $z = \pm\omega_0(\beta)$. These poles correspond to purely oscillatory modes which, as we shall see, correspond to incoherent bubble motion. In particular one can prove, using Laplace transform techniques (see Ref. 1), that

$$\hat{\hat{r}} \approx \hat{\hat{r}}_I + \hat{\hat{r}}_C, \tag{5.43}$$

where

$$\hat{\hat{r}}_I = A_I \, \mathrm{e}^{\mathrm{i}\omega_0(\beta)t} + B_I \, \mathrm{e}^{-\mathrm{i}\omega_0(\beta)t} \tag{5.44}$$

and

$$\hat{\hat{r}}_C = \mathrm{e}^{-\delta t} \left(A_C \, \mathrm{e}^{\mathrm{i}\Omega t} + B_C \, \mathrm{e}^{i\Omega t} \right), \tag{5.45}$$

where δ and Ω have the same values as given in (5.41). A_I, A_C, B_I, and B_C are constants that depend on the initial conditions and the bubble size distribution.

We notice that the first term of (5.43) has no damping term and the frequency of oscillation depends on the bubble size. This solution thus corresponds to bubbles which ultimately oscillate completely out of phase with each other. To help the reader understand the process of the bubble motion becoming incoherent we plot $\hat{\hat{r}} = \sin(\omega_0(\beta)t)$ for five different values of β on the same graph (see figure 3). Initially, the bubble motion is coherent but as time progresses the motion becomes increasingly incoherent because the oscillation frequency for bubbles of different sizes is different. Since there is a size distribution of bubbles it follows then that nearby bubbles will oscillate incoherently. On the other hand, the frequency of the second term of (5.43) does not depend on the bubble size and consequently corresponds to coherent bubble oscillations.

If we choose our bubble size distribution to be $g_\varepsilon(\beta)$ it is possible then to prove that $\lim_{\varepsilon\to 0} A_I = 0$ and $\lim_{\varepsilon\to 0} B_I = 0$. This is done by analysing the residues of the poles located at $z = \pm\omega_0(\beta)$. It also follows from (5.40) that $\lim_{\varepsilon\to 0} \delta = 0$. This indicates that if we consider the case when all of the bubbles have the same size, the strength of the incoherent bubble oscillations is zero and the pressure waves will no longer damp.

Thus we see that even though the pressure perturbation will decay to zero the bubbles will continue to oscillate. As the pressure decays the bubble oscillations become completely incoherent. In other words the bubble oscillations are all completely out of phase with each other and they make no net contribution to the pressure. Hence we conclude that sound is the result of coherent bubble motion.

5.3.1. *The dispersion relation revisited*

In the analysis in § 5.2 we considered the initial value problem and took k to be real. Let us now relax this assumption and look for solutions of the form $\exp(\mathrm{i}(kx - \Omega t))$, where Ω is real and k can be complex. It follows from the arguments in § 5.2 that we will have solutions of this form provided

$$\lim_{v\to 0^+} D(\Omega + \mathrm{i}v) = 0. \tag{5.46}$$

Since $D(z) = 0$ implies that

$$k^2 = 3\chi \int \frac{z^2 \beta g(\beta)}{\omega_0(\beta) - z^2} \, \mathrm{d}\beta$$

we can use (5.30) along with (5.31) and (5.32) to show that (5.46) is equivalent to

$$k^2 = 3\chi \fint \frac{\Omega^2 \beta g(\beta)}{\omega_0(\beta)^2 - \Omega^2}\,\mathrm{d}\beta + \mathrm{i}\frac{\pi \mathrm{sgn}(\Omega) 3\chi\Omega^2 \beta_\Omega g(\beta_\Omega)}{2\omega_0(\beta_\Omega)|\omega_0'(\beta_\Omega)|}.$$

Using the expressions for $\omega_0(\beta)$ and β_Ω, the above equation can be written as

$$\begin{aligned} k^2 &= 3\chi \fint \frac{\Omega^2 \beta g(\beta)}{\omega_0(\beta)^2 - \Omega^2}\,\mathrm{d}\beta + \mathrm{i}\frac{\pi}{2}\,\mathrm{sgn}(\Omega)\,3\chi\beta_\Omega^2 g(\beta_\Omega) \\ &= A + \mathrm{i}\,\mathrm{sgn}(\Omega) B. \end{aligned}$$

This is how one can obtain (5.11).

6. Nonlinear dynamics

We have seen that even in the absence of damping mechanisms for bubble motion the pressure waves will be damped when there is a size distribution of bubbles. Since bubbles of different sizes can oscillate with different frequencies, they eventually will oscillate completely out of phase with each other. These incoherent bubble oscillations produce no net pressure field and as a consequence the pressure field decays to zero even though the bubbles continue to oscillate.

It is known that the frequency of a bubble oscillation depends on its amplitude (see, for example, Prosperetti 1975 or Smereka *et al.* 1987). Hence bubbles with a distribution of amplitudes could give rise to incoherent oscillations. If nonlinear effects are considered it is possible to observe a similar damping mechanism as described in § 5 for bubbles of identical sizes. We will show that this is indeed the case.

We imagine the following idealized situation: a wave breaks and produces a cloud of bubbles with identical equilibrium sizes but with a distribution of different phases and amplitudes. Furthermore, we shall consider the situation with an ideal bubbly fluid. There are two reasons for restricting the analysis to this idealized problem. First, it will delineate any damping that will arise from nonlinear effects; we know that for an ideal bubbly fluid with no size distribution of bubbles pressure waves will not be damped if one only considers linear effects. Second, because the nonlinear case is technically much more difficult, it appears that we can only make progress by relying on the Hamiltonian form of the equations. It should be noted that the Hamiltonian form has been useful in other problems related to bubble dynamics, see for example, Benjamin (1987), Yurkovetsky & Brady (1996), and Russo & Smereka (1996).

Our goal in this section is to study the initial value problem for the situation described above when the initial condition consists of a cloud of bubbles where the amplitudes and phases are completely incoherent, plus a small perturbation which has a coherent component. The incoherent part is defined to be that which causes no pressure waves. We shall see that the pressure waves damp to zero as the bubble oscillations become incoherent. We begin by outlining the Hamiltonian form of the equations of motion and explain in more detail the incoherent component.

6.1. *Hamiltonian form*

The nonlinear case is much more difficult than the preceding linear case and we have found it useful to express the equations in Hamiltonian form. In Appendix B we show that the Hamiltonian version of (4.7) is

$$\frac{\partial f}{\partial t} + w\frac{\partial f}{\partial r} + F\frac{\partial f}{\partial z} = 0 \tag{6.1}$$

with

$$w = \frac{\partial H}{\partial z} \quad \text{and} \quad F = -\frac{\partial H}{\partial w},$$

where H is the Hamiltonian and is given by (B 13). As we have done in previous sections, we consider a spatially homogeneous cloud of bubbles of infinite extent; hence we have $V \to \infty$ and $\eta(\boldsymbol{x}) = 1$.

Since the bubble radius must be positive it follows that f must be zero for $r < 0$. We also note that

$$\int_{z=-\infty}^{\infty} \int_{r=0}^{\infty} \frac{z}{r} f_0(h(r,z))\, \mathrm{d}r\, \mathrm{d}z = 0 \tag{6.2}$$

for any f_0. We assume that $f_0(x)$ vanishes sufficiently fast as $x \to \infty$ so the above integral converges. Equation (6.2) is true because $f_0(h(r,z))$ is an even function of z while z/r is odd, consequently their product integrates to zero. This implies for $f(r,z,\boldsymbol{x},t) = f_0(h(r,z))$ that $q(\boldsymbol{x},t) = 0$. Therefore we find from (B 10) and (B 11) that

$$w = \frac{z}{r^3} \quad \text{and} \quad F = \frac{3z^2}{2r^4} - u'(r)$$

and a straightforward computation shows that

$$\left(w\frac{\partial}{\partial r} + F\frac{\partial}{\partial z} \right) f_0(h(r,z)) = 0.$$

Thus $f_0(h(r,z))$ is a solution of (6.1).

Each level curve of $h(r,z)$ corresponds to a one parameter family of periodic bubble oscillations which are parameterized by the phase. Different level curves correspond to bubble oscillations of different amplitudes. Therefore we see that $f_0(h(r,z))$ corresponds to a collection of bubbles oscillating with different amplitudes and phases. The net interaction is cancelled and they oscillate as if they were isolated from each other.

Using the results in Appendix B, one can show that the perturbed pressure can be written as

$$\phi(\boldsymbol{x},t) = 3\chi \frac{\partial}{\partial t} Q_R^{-1}(q), \tag{6.3}$$

where $q = \int z/rf(r,z,\boldsymbol{x},t)\, \mathrm{d}r\, \mathrm{d}z$ and Q_R is defined in Appendix B (see (B 9)). It follows from (6.2) and (6.3) that the perturbed pressure field is zero if $f = f_0(h(r,z))$. Thus we see that when bubbles are desynchronized the perturbed pressure field is zero. Furthermore any synchronization of the bubbles will cause the production of sound.

As pointed out in Ref. 1, the initial value problem is best studied in action-angle variables for the single bubble; in these coordinates the kinetic equation given by (6.1) becomes

$$\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \theta} + \Upsilon \frac{\partial f}{\partial I} = 0. \tag{6.4}$$

The expressions for Ω and Υ are given in Appendix B. Most importantly, the incoherent solution will be $f_0(I)$.

6.2. *Initial value problem*

Our goal is to study the initial value problem for (6.4) with general initial conditions. Unfortunately this is too complex and here we will instead consider the situation where the initial conditions are given by

$$f(\theta, I, \boldsymbol{x}, 0) = f_0(I) + g_0(\theta, I, x) \quad \text{where} \quad g_0 \ll 1. \tag{6.5}$$

The first term corresponds to a spatially uniform bubble mixture where the bubbles have random phases and amplitudes; this gives rise to incoherent bubble motion. The second term corresponds to a spatially varying perturbation. To study the initial value problem with (6.5) we substitute

$$f(\theta, I, \boldsymbol{x}, t) = f_0(I) + g(\theta, I, x, t)$$

into (6.4). We take $g \ll 1$ and neglect terms of $O(g^2)$. Next we consider

$$\hat{g}(\theta, I, k, t) = \int \mathrm{e}^{\mathrm{i}kx} g(\theta, I, x, t)\,\mathrm{d}x$$

and arrive at the following equation for $\hat{g}$:

$$\frac{\partial \hat{g}}{\partial t} + \omega(I)\frac{\partial \hat{g}}{\partial \theta} + \Upsilon f_0'(I) = 0, \tag{6.6}$$

where

$$\left.\begin{aligned} \Upsilon &= \frac{6\pi\chi\hat{q}(k,t)}{k^2 + 3\chi\overline{R_0}}\frac{\partial \mathscr{S}}{\partial \theta}, \\ \hat{q}(k,t) &= \frac{1}{2\pi}\int \mathscr{S}(\theta, I)\,\hat{g}(\theta, I, k, t)\,\mathrm{d}I\,\mathrm{d}\theta, \\ \overline{R_0} &= \int \mathscr{R}(I,\theta) f_0(I)\,\mathrm{d}I\,\mathrm{d}\theta \quad \text{and} \quad \mathscr{S}(I,\theta) = \frac{\mathscr{Z}(I,\theta)}{\mathscr{R}(I,\theta)}. \end{aligned}\right\} \tag{6.7}$$

Before continuing with the analysis, let us point out that even though (6.4) is linearized the bubble dynamics are still nonlinear. Also, we note that if we linearize (6.3), take the Fourier transform, use (6.6), and integrate by parts one can show that

$$\hat{\phi} = \frac{3\chi}{k^2 + 3\chi}\int \mathscr{S}_\theta \omega(I)\hat{g}\,\mathrm{d}I\,\mathrm{d}\theta. \tag{6.8}$$

Thus perturbations of the incoherent initial condition will give rise to a pressure disturbance. The main result of the section proves that

$$\hat{\phi} \approx \mathrm{e}^{-\delta t}(A_p\,\mathrm{e}^{\mathrm{i}\Omega t} + B_p\,\mathrm{e}^{-\mathrm{i}\Omega t}),$$

where Ω is given by (2.8). If we compare this result with (2.10), we see that the incoherent bubble oscillations cause $\hat{\phi}$ to decay.

This result is obtained by first observing that the equation for $\hat{g}$ is exactly the same as the equation obtained in Ref. 1. This indicates that each Fourier mode of $\hat{g}$ behaves as a globally coupled oscillator (we have included a $1/2\pi$ factor in the expression for $\hat{q}(k,t)$ to conform more closely with the notation used in Ref. 1). In order to make the following analysis rigorous a number of technical assumptions need to made about f_0 and $\hat{g}$. For brevity these will not be stated here, the reader is referred to Ref. 1.

Next $\hat{g}$ and $\mathscr{S}$ are written as a Fourier series; we have

$$\hat{g}(\theta, I, k, t) = \sum_{m=-\infty}^{\infty} g_m(I, k, t)\,\mathrm{e}^{\mathrm{i}m\theta} \quad \text{and} \quad \mathscr{S}(\theta, I) = \sum_{m=-\infty}^{\infty} s_m(I)\,\mathrm{e}^{\mathrm{i}m\theta}.$$

Therefore (6.6) becomes

$$\frac{\partial g_m}{\partial t} + \mathrm{i}m\omega(I)g_m + \frac{6\pi\chi m\mathrm{i}}{k^2 + 3\chi\overline{R_0}} s_m(I) f_0'(I)\hat{q}(k,t) = 0, \tag{6.9}$$

where

$$\hat{q}(k,t) = \sum_{n=-\infty}^{\infty} \int_0^{\infty} g_{-n}(I,k,t) s_n(I)\,\mathrm{d}I.$$

We note that $s_0(I) = 0$. Using Laplace transform techniques (see Ref. 1 for details) we find

$$\widehat{q}(k,t) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty+\mathrm{i}\Lambda}^{\infty+\mathrm{i}\Lambda} \frac{T(z)}{D(z)} \mathrm{e}^{-\mathrm{i}zt}\,\mathrm{d}z. \tag{6.10}$$

In the above integral, Λ is sufficiently large so that the path of integration is above any singularities of the integrand. The functions in the integrand are defined as

$$T(z) = \sum_{m=1}^{\infty} T_m(z) \quad \text{with} \quad T_m(z) = \int_0^{\infty} \left(\frac{s_{-m}(I) g_m(I,k,0)}{m\omega(I) - z} - \frac{s_m(I) g_{-m}(I,k,0)}{m\omega(I) + z} \right) \mathrm{d}I$$

and

$$D(z) = 1 + \frac{6\pi\chi}{k^2 + 3\chi \overline{R}_0} \sum_{m=1}^{\infty} D_m(z) \quad \text{with} \quad D_m(z) = \int_0^{\infty} \frac{2m^2 s_m(I) s_{-m}(I) f_0'(I) \omega(I)\,\mathrm{d}I}{m^2\omega^2(I) - z^2}. \tag{6.11}$$

Note that the dependence of both D and T on k has been suppressed for clarity.

It is a simple matter to verify with the assumptions we have made that both $T(z)$ and $D(z)$ are analytic functions, provided $\mathrm{Im}(z) \neq 0$. We will now assume that $D(z)$ has only simple zeros. Therefore the integrand of (6.10) is meromorphic with poles located at zeros of $D(z)$. Since $D(\bar{z}) = \bar{D}(z)$, if z is a zero then $\bar{z}$ is also a zero. Furthermore, $-z$ and $-\bar{z}$ will also be zeros since $D(z) = D(-z)$. Let $z_m, m = 1$, to M denote the roots of $D(z) = 0$ with $\mathrm{Im}(z_m) \neq 0$. It is shown in Ref. 1 that we can write $\hat{q}(k,t)$ as

$$\hat{q}(k,t) = -\sum_{m=1}^{M} \frac{T(z_m)}{D'(z_m)} \mathrm{e}^{-\mathrm{i}z_m t} + \int_{-\infty}^{\infty} J(u)\,\mathrm{e}^{-\mathrm{i}ut}\,\mathrm{d}u, \tag{6.12}$$

where $2\pi \mathrm{i} J(u)$ denotes the jump of T/D across the u-axis. It is evident that if $D(z)$ has any zeros with $\mathrm{Im}(z) \neq 0$ then $\hat{q}(k,t)$ must grow exponentially in time. On the other hand, if $D(z)$ does not have any zeros it then follows that

$$\hat{q}(k,t) = \int_{-\infty}^{\infty} J(u)\,\mathrm{e}^{-\mathrm{i}ut}\,\mathrm{d}u. \tag{6.13}$$

It is shown in Ref. 1 that $J(u)$ is a L^1 function. It follows therefore from the Riemann–Lesbegue lemma that

$$\hat{q}(k,t) \to 0 \quad \text{as} \quad t \to \infty. \tag{6.14}$$

It is also shown in Ref. 1 that $g \to 0$ (weakly).

To summarize, we have shown that the incoherent state is asymptotically stable if $D(z)$ has no zeros. In particular this means, from Parseval's Theorem, that

$$\int_{-\infty}^{\infty} |q|^2\,\mathrm{d}x \to 0 \quad \text{as} \quad t \to \infty.$$

In view of (6.3) this indicates that the energy in the sound field, $E_S(t)$, will also decay to zero.

It is evident from the previous discussion that the desynchronized solution is linearly unstable if $D(z)$ has zeros with $\mathrm{Im}(z) > 0$ and is linearly stable if $D(z)$ has no zeros or only zeros with $\mathrm{Im}(z) = 0$. The form of $D(z)$ is rather complex and furthermore we do not have explicit expressions for $s_m(I)$ and $\omega(I)$. This makes it difficult to determine whether or not $D(z)$ has any zeros. Nevertheless, the following situation produces definite results; let

$$f_0(I) = \frac{1}{2\pi\sigma} f_s(I/\sigma), \tag{6.15}$$

where $f_s(I)$ has a single maximum at $I = 0$ and tends to zero as $I \to \infty$. In addition

$$\int_0^\infty f_s(x)\,\mathrm{d}x = 1 \quad \text{and} \quad f_s > 0;$$

the limit $\sigma \to 0$ corresponds to the bubbles being at rest.

Using the argument principle, as outlined in Appendix A, we may prove that $D(z)$ has no zeros provided that $D_R(0) > 0$. Below we shall prove that

$$D(0) = \frac{k^2}{k^2 + 3\chi\overline{R_0}}. \tag{6.16}$$

Therefore it follows that $D_R(0) > 0$ provided $k \neq 0$. Equation (6.16) indicates that $D = 0$ for $k = 0$. Thus the initial value problem has a zero eigenvalue. This corresponds to a perturbation which takes the solution to a nearby incoherent state. Therefore we shall no longer consider the $k = 0$ mode.

To verify (6.16) we evaluate D at $z = 0$ using (6.11) to obtain

$$D(0) = 1 + \frac{12\pi\chi}{k^2 + 3\chi\overline{R_0}} \sum_{m=1}^{\infty} \int_0^\infty \frac{s_m(I)s_{-m}(I)f_0'(I)}{\omega(I)}\,\mathrm{d}I. \tag{6.17}$$

It follows from Parseval's Theorem that

$$\sum_{m=1}^{\infty} s_m(I)s_{-m}(I) = \frac{1}{4\pi}\int_0^{2\pi} \mathscr{S}^2(I,\theta)\,\mathrm{d}\theta \equiv \frac{N(I)}{4\pi}. \tag{6.18}$$

Next we substitute (6.18) into (6.17), integrate by parts and use the definition of $\overline{R_0}$ to obtain

$$D(0) = \frac{k^2 + 3\chi \displaystyle\int_0^\infty \left[R_0(I) - \frac{\mathrm{d}}{\mathrm{d}I}\left(\frac{N(I)}{\omega(I)}\right)\right] f_0(I)\mathrm{d}I}{k^2 + 3\chi\overline{R_0}}, \tag{6.19}$$

where $R_0(I) = \int_0^{2\pi} R(I,\theta)\mathrm{d}\theta$. It is shown in Appendix C that

$$R_0(I) = \frac{\mathrm{d}}{\mathrm{d}I}\left(\frac{N(I)}{\omega(I)}\right). \tag{6.20}$$

Thus (6.16) follows from (6.19) and (6.20).

We have now established that $D(z)$ has no zeros when $f_0(I)$ is given by (6.15); thus $\hat{q}(k,t)$ is given by (6.13). Following a procedure similar to that used in §5, we can show that $\hat{q}(k,t)$ will also exponentially decay to zero. We begin with

$$\lim_{v\to 0^+} D(u \pm \mathrm{i}v) = D_R(u) \pm \mathrm{i}D_I(u), \tag{6.21}$$

where

$$D_R(u) = 1 + \frac{6\pi\chi}{k^2 + 3\chi\overline{R_0}} \sum_{m=1}^{\infty} D_{m,R}(u) \quad \text{and} \quad D_I(u) = \frac{6\pi\chi}{k^2 + 3\chi\overline{R_0}} \sum_{m=1}^{\infty} D_{m,I}(u) \tag{6.22}$$

with

$$D_{m,R}(u) = \fint_0^\infty \frac{2m^2 s_m(I) s_{-m}(I)\omega(I) f_0'(I)\,\mathrm{d}I}{m^2\omega^2(I) - u^2} \tag{6.23}$$

and

$$D_{m,I}(u) = \begin{cases} \dfrac{\pi \mathrm{sgn}(u) s_m(I_u) s_{-m}(I_u) f_0'(I_u)}{|\omega'(I_u)|}, & |u| < m\omega_0 \\ 0, & \text{otherwise,} \end{cases} \tag{6.24}$$

where I_u is the single positive root of the equation

$$m\omega(I_u) = |u|. \tag{6.25}$$

The above results can be found in Ref. 1. As before we can now compute $J(u)$ and find that $d(u)$ can become small. Using results in Appendix D we have

$$D_R(u) = 1 - \frac{3\chi}{k^2 + 3\chi}\left(\frac{\omega_0^2}{\omega_0^2 - u^2} + O(\sigma)\right). \tag{6.26}$$

It follows directly from (6.26) that $D_R(\pm\Omega) = 0$, where

$$\Omega = \frac{\omega_0 k}{\sqrt{3\chi + k^2}} + O(\sigma). \tag{6.27}$$

In Appendix E it is shown that $D_I(\Omega) \to 0$ as $\sigma \to 0$ and it follows that $D_I(\Omega) \ll 1$ if $\sigma \ll 0$. Following a similar argument as found in § 5 we find

$$\hat{q}(k,t) \propto \mathrm{e}^{-\delta t}\left(A_q\,\mathrm{e}^{\mathrm{i}\Omega t} + B_q\,\mathrm{e}^{-\mathrm{i}\Omega t}\right) \quad \text{as} \quad \sigma \to 0 \quad \text{provided} \quad t = O(\delta^{-1}) \tag{6.28}$$

where

$$\delta = \frac{3\chi\omega_0 D_I(\Omega)}{2k\sqrt{3\chi + k^2}} \tag{6.29}$$

and Ω is given by (6.27). A_q and B_q are constants that depend on the incoherent state and the perturbation.

The above results, when combined with the linear version of (6.3), allow us to conclude that for $\sigma \ll 1$ the Fourier transform of the pressure waves has the following behaviour:

$$\hat{\phi} = \mathrm{e}^{-\delta t}\left(A_p\,\mathrm{e}^{\mathrm{i}\Omega t} + B_p\,\mathrm{e}^{-\mathrm{i}\Omega t}\right) \quad \text{if} \quad t = O(\delta^{-1}). \tag{6.30}$$

We notice that as $\sigma \to 0$ (6.30) will approach (2.10) as one would expect. This calculation shows that even though the bubbles have identical sizes, the pressure waves will still damp to zero. This is because the bubble oscillations will again become incoherent. The incoherence of the bubble oscillations in this situation arises from the fact that the bubble oscillation frequency depends on its amplitude. Therefore if nearby bubbles are given different initial conditions nonlinear effects will cause them to oscillate at different frequencies. This will give rise to incoherent bubble oscillations.

7. Effects of viscosity and heat transfer

In this section we examine the effects of bubble damping mechanisms. The primary damping mechanisms are liquid viscosity, heat transfer, and acoustic radiation. For an excellent review of damping mechanisms see van Wijngaarden (1972). In this paper we only include effects of liquid viscosity and heat transfer. If these effects are included into the equation of motion for a single bubble, one finds

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 + 4\mu\frac{\dot{R}}{R} = \frac{1}{\rho_\ell}(P - P_\infty), \tag{7.1}$$

where P is the gas pressure and μ is the liquid viscosity. It follows from conservation of energy that

$$\dot{P} = \frac{3(\gamma-1)Q}{4\pi R^3} - \frac{3\gamma\dot{R}}{R}P, \tag{7.2}$$

where Q is the amount of heat entering the bubble from the liquid.

We compute Q using a model due to Gubaidullin *et al.* (1976, but see also Nigmatulin 1982) which is

$$Q = 4\pi R^2 K\, Nu\frac{(T_\infty - T_g)}{2R}, \tag{7.3}$$

where K is the thermal conductivity of the gas phase, T_∞ is the ambient liquid temperature, T_g is the temperature of the gas bubble, and Nu is the heat transfer coefficient between the gas and the liquid. The temperature of the gas bubble will be determined by the ideal gas law, therefore we have

$$T_g = T_\infty\frac{P}{P_\infty}\left(\frac{R}{R_0}\right)^3. \tag{7.4}$$

Although it is a crude approximation, we shall consider Nu to be constant. If $Nu = 0$ the bubble motion is adiabatic and in the limit $Nu \to \infty$ the bubble motion is isothermal. More sophisticated models for heat transfer can be found in Chapman & Plesset (1971), Miksis & Ting (1984), Prosperetti, Crum & Commander (1988), and Prosperetti (1991) for example. These could also be incorporated into a model of this form. The effects of acoustic radiation can be included in a similar fashion using the work of Commander & Prosperetti (1989). This effect gives rise to an additional damping mechanism.

The equations of motion for the N interacting bubble in dimensionless variables are

$$r_k\ddot{r}_k + \tfrac{3}{2}\dot{r}_k^2 + \mu\frac{\dot{r}_k}{r_k} = p_k - 1 - \phi_k,$$

where

$$\phi_k = \frac{3\kappa}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\neq k}\frac{r_j^2\dot{r}_j}{4\pi|\boldsymbol{x}_k - \boldsymbol{x}_j|}.$$

The dimensionless gas pressure in the kth bubble is denoted p_k and is determined from

$$\dot{p}_k = \frac{\zeta}{r_k^2}\left(1 - \frac{p_k r_k^3}{\beta_k}\right) - \frac{3\gamma p_k\dot{r}_k}{r_k}.$$

This is the dimensionless form of (7.2). The dimensionless constants used above are

$$\mu = \frac{4\mu_\ell}{\langle R_0\rangle\sqrt{\rho_\ell P_\infty}} \quad \text{and} \quad \zeta = \frac{3(\gamma-1)K\,Nu\,T_\infty}{2P_\infty\langle R_0\rangle}\sqrt{\frac{\rho_\ell}{P_\infty}}.$$

For the development of our kinetic equation for the above system we proceed in the same fashion as in §4. First we write our equations as a first-order system

$$\dot{r}_k = w_k, \quad \dot{w}_k = a_k \quad \text{and} \quad \dot{p}_k = b_k,$$

where

$$a_k = r_k^{-1}\left[-\tfrac{3}{2}w_k^2 + p_k - 1 - \phi_k - \mu\frac{w_k}{r_k}\right]$$

and

$$b_k = \frac{\zeta}{r_k^2}\left(1 - \frac{p_k r_k^3}{\beta_k}\right) - \frac{3\gamma p_k w_k}{r_k}.$$

Next we consider the density function

$$\rho_N(r, w, p, \beta, \boldsymbol{x}, t) = \frac{V^*}{N}\sum_{k=1}^{N} \delta(r - r_k(t))\delta(w - w_k(t))\delta(p - p_k(t))\delta(\beta - \beta_k)\delta(\boldsymbol{x} - \boldsymbol{x}_k).$$

We compute $\partial_t \rho_N$ and let $N \to \infty$ to obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial (w\rho)}{\partial r} + \frac{\partial (a_1 \rho)}{\partial w} + \frac{\partial (a_2 \rho)}{\partial p} = 0, \tag{7.5}$$

where

$$a_1 = r^{-1}\left[-\tfrac{3}{2}w^2 + p - 1 - \phi(\boldsymbol{x}, t) - \mu\frac{w}{r}\right]$$

and

$$a_2 = \frac{\zeta}{r^2}\left(1 - \frac{pr^3}{\beta^3}\right) - \frac{3\gamma p w}{r}.$$

The sound field is given by

$$-\Delta\phi + 3\chi\eta(\boldsymbol{x})R(\boldsymbol{x}, t)\phi = 3\chi B(\boldsymbol{x}, t), \tag{7.6}$$

where

$$\eta(\boldsymbol{x})R(\boldsymbol{x}, t) = \int r\rho(r, w, p, \beta, \boldsymbol{x}, t)\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}p\,\mathrm{d}\beta$$

and

$$B(\boldsymbol{x}, t) = \int \left(\tfrac{1}{2}w^2 + p - 1 - \mu\frac{w}{r} + vrb\right) r\rho(r, w, p, \beta, \boldsymbol{x}, t)\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}p\,\mathrm{d}\beta.$$

7.1. *Linear dynamics*

In this section we shall compare our model with the work of Carstensen & Foldy (1947) and Commander & Prosperetti (1989). To this end we must first linearize the discrete equations of motion. We let $r_k = \beta_k + r_k'$ and $p_k = 1 + p_k'$, assume $r_k' \ll 1$, $p_k' \ll 1$ and $w_k \ll 1$ and follow steps similar to those used in § 5 to obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial (w\rho)}{\partial r} + \frac{\partial (a_1 \rho)}{\partial w} + \frac{\partial (a_2 \rho)}{\partial p} = 0, \tag{7.7}$$

where

$$a_1 = \frac{1}{\beta}\left(p - \phi - \frac{4\mu w}{\beta}\right), \quad a_2 = -\frac{\zeta}{\beta^2}\left(p + \frac{3r}{\beta}\right) - \frac{3\gamma w}{\beta}, \quad -\Delta\phi + 3\chi\phi = -3\chi B(\boldsymbol{x}, t),$$

and

$$B(\boldsymbol{x}, t) = -\int \beta^2\omega_0^2(\beta) r\rho\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}p\,\mathrm{d}\beta.$$

Next we consider the phase-averaged quantities $\bar{r}, \bar{w}$, and $\bar{p}$ where

$$\bar{r} = \bar{r}(\boldsymbol{x}, t, \beta) = \frac{\int r\rho\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}p}{\int \rho\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}p} = \frac{1}{g(\beta)}\int r\rho\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}p.$$

$\bar{w}$ and $\bar{p}$ are defined in a similar fashion. By taking moments of (7.7), we find the time evolution of $\bar{r}$ and $\bar{p}$ to be

$$\frac{\partial^2 \bar{r}}{\partial t^2} - \frac{\bar{p}}{\beta} + \frac{\phi}{\beta} + \frac{\mu \bar{w}}{\beta^2} = 0 \tag{7.8}$$

and

$$\frac{\partial \bar{p}}{\partial t} + \frac{\zeta}{\beta^2}\left(\bar{p} + \frac{3\bar{r}}{\beta}\right) + \frac{3\gamma \bar{w}}{\beta} = 0. \tag{7.9}$$

The expression for ϕ can be expressed in terms of $\bar{r}$ and we have

$$-\Delta\phi = 3\chi \int \beta^2 \frac{\partial^2 \bar{r}}{\partial t^2} g(\beta)\, \mathrm{d}\beta. \tag{7.10}$$

Equations (7.8), (7.9), and (7.10) are essentially the effective equations in dimensionless form derived by Commander & Prosperetti (1989) in the incompressible limit ($C_\ell \to \infty$). The only difference here is that we use a simpler model for thermal effects.

7.2. *Dispersion relation*

In this section we compute the dispersion relation for the set effective equations derived in the previous section. We look for solutions of the form $\mathrm{e}^{\mathrm{i}(kx-\Omega t)}$ for (7.8), (7.9), and (7.10) and find the following dispersion relation:

$$k^2 = 3\chi \int \frac{\beta g(\beta)\, \mathrm{d}\beta}{\omega_0^2(\beta) - \Omega^2 - 2ib\Omega}, \tag{7.11}$$

where

$$\omega_0^2(\beta, \Omega) = \frac{1}{\beta^2}\mathrm{Re}(\Phi), \quad b(\beta, \Omega) = \frac{2\mu}{\beta^2} + \frac{1}{2\Omega\beta^2}\mathrm{Im}(\Phi)$$

and

$$\Phi = 3\gamma \left(\frac{1 - \mathrm{i}\zeta/(\Omega\beta^2\gamma)}{1 - \mathrm{i}\zeta/(\Omega\beta^2)} \right).$$

This form of the dispersion relation was previously derived by Commander & Prosperetti (1989). They included effects of liquid compressibility which we have ignored here and also used a different model for thermal effects. In addition they evaluated this dispersion relation numerically and performed careful comparisons with experiments. In most cases the agreement was quite good with the exception of frequencies close to resonance.

Sangani (1991) calculated the dispersion relation for a mixture of bubbles with identical equilibrium sizes. He included liquid compressibility, thermal effects, liquid viscosity and surface tension. He also included higher-order terms in the volume fraction. If we restrict (7.11) to the case with one bubble size and ignore liquid compressibility, surface tension, and higher effects of volume fraction then (7.11) will be in agreement with the results of Sangani (note one slight difference: the Φ function is different owing to the fact that we use a different model for thermal effects).

7.3. *Initial value problem*

In this section we consider the initial value problem for the effective equations derived above ((7.8), (7.9), and (7.10)). This is a rather complex problem so we restrict ourselves to the case when $b \ll 1$ and $\varepsilon \ll 1$ as it simplifies the analysis somewhat. We note that $b \ll 1$ provided $\mu \ll 1$ and $\zeta \ll 1$ or $\zeta \gg 1$. We follow the same steps as in § 5.2

and find that $\hat{\phi}$ is given by (5.24), where $D(z)$ is now

$$D(z) = k^2 - 3\chi \int_0^\infty \frac{z^2 \beta g(\beta)}{\omega_0^2(\beta, z) - z^2 - 2\mathrm{i}zb(\beta, z)} \mathrm{d}\beta. \tag{7.12}$$

A similar expression can be computed for $T(z)$ but it will not be needed. We shall not follow the approach used in § 5.2 to analyse the behaviour of $\hat{\phi}$ since the $2\mathrm{i}zb$ term in the denominator complicates the analysis. Instead we shall evaluate $D(z)$ on the real axis (both $D(z)$ and $T(z)$ are now analytic on the real axis) and then verify that $D(z)$ has two zeros in the lower-half of the complex z-plane.

We apply the same ideas as found in Appendix D to show that

$$D_R(u) = k^2 - \frac{3\chi u^2}{\omega_0^2(1, u) - u^2} \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad b \to 0. \tag{7.13}$$

We can show that

$$D_I(u) = -\frac{3\chi\pi\beta_u g(\beta_u) u}{2|W|} \quad \text{as} \quad b \to 0, \tag{7.14}$$

where β_u is the resonant bubble size satisfying $\omega_0(\beta_u, u) = u$ and $W = \partial\omega(\beta, u)/\partial\beta|_{\beta=\beta_u}$. We also find

$$D_I(u) = -\frac{6\chi b(1, u) u^3}{(\omega_0^2(1, u) - u^2)^2 + 4u^2 b^2(1, u)} \quad \text{as} \quad \varepsilon \to 0. \tag{7.15}$$

Combining (7.14) and (7.15), we find that when $b \ll 1$ and $\varepsilon \ll 1$

$$D_I(u) = -\frac{3\chi\pi\beta_u g(\beta_u) u}{2|W|} - \frac{6\chi b(1, u) u^3}{(\omega_0^2(1, u) - u^2)^2} \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad b \to 0. \tag{7.16}$$

Next we use (7.13) and (7.16) to show that $D(z^*) = 0$ with $z^* \approx \pm\Omega - \mathrm{i}\delta$, where

$$D_R(\Omega) = 0$$

and

$$\delta = \left| \frac{D_I(\Omega)}{D_R'(\Omega)} \right|. \tag{7.17}$$

Since $D(z)$ has zeros at $z^* = \pm\Omega - i\delta$ it then follows, from Laplace transform techniques, that

$$\hat{\phi} \approx \mathrm{e}^{-\delta t} \left(A_p \mathrm{e}^{\mathrm{i}\Omega t} + B_p \mathrm{e}^{-\mathrm{i}\Omega t} \right).$$

Once again we observe that the Fourier transform of the pressure waves both oscillates and decays. It follows from (7.16) and (7.17) that the damping now has two components, one due to the size distribution and the other from damping effects associated with single-bubble dynamics. The expression for the damping term given by (7.16) and (7.17) is rather complex and it is useful to consider a special case to further elucidate it. As pointed out by Prosperetti (1991), the nearly isothermal situation is the most important case when considering wave propagation in a bubbly fluid. In our model the nearly isothermal case occurs when $\zeta \gg 1$. In this situation one has

$$\omega_0^2 \approx \frac{3}{\beta^2} \quad \text{and} \quad b \approx \frac{2\mu}{\beta^2} + \frac{3(\gamma - 1)}{2\zeta}.$$

Using the above formulae in (7.13), (7.16) and (7.17), we find that

$$\Omega = \frac{|k|\sqrt{3}}{\sqrt{k^2 + 3\chi}} \tag{7.18}$$

and

$$\delta \approx \frac{(3\chi)^2\pi}{4|k|^3}\sqrt{\frac{3}{k^2+3\chi}}g\left(\frac{\sqrt{k^2+3\chi}}{|k|}\right)+\left(2\mu+\frac{3(\gamma-1)}{2\zeta}\right)\frac{k^2}{k^2+3\chi}. \tag{7.19}$$

It follows from (7.19) that the pressure waves will damp from both incoherent bubble oscillations and single-bubble damping effects.

We also compute the bubble behaviour using a similar approach and find

$$\hat{\hat{r}} \approx \hat{\hat{r}}_I + \hat{\hat{r}}_C, \tag{7.20}$$

where

$$\hat{\hat{r}}_I = \mathrm{e}^{-\delta_R t}\left(A_I \mathrm{e}^{\mathrm{i}\omega_0(\beta)t} + B_I \mathrm{e}^{-\mathrm{i}\omega_0(\beta)t}\right),$$
$$\hat{\hat{r}}_C = \mathrm{e}^{-\delta t}\left(A_C \mathrm{e}^{\mathrm{i}\Omega t} + B_C \mathrm{e}^{-\mathrm{i}\Omega t}\right)$$

and $\delta_R = 2\mu + 2(\gamma - 1)/2(\zeta)$. δ and Ω are given by (7.19) and (7.18). A_I, A_C, B_I, and B_C are constants that depend on the initial conditions and the bubble size distribution. Following the same line of reasoning as in §5.2 we observe that the first term of (7.20) corresponds to incoherent bubble oscillations whereas the second term corresponds to coherent oscillations. If we compare δ and δ_R we see that for bubbly fluids with a large spread of bubble sizes the pressure waves can damp faster than the bubble oscillations.

Finally, let us mention the application of this model to periodically forced bubble clouds. If the forcing is small then the bubble motion will be approximated well by linear motion and we can use the model given by (7.7) with boundary conditions $\phi(0,t) = \phi(L,t) = A\sin\Omega t$. In this case, it is not hard to prove that for large times bubble motion becomes completely coherent for all initial conditions. The periodic solution that results is similar to that described in the work of Smereka & Banerjee (1988). However if the forcing is large then one must use (7.5); in this case it is possible that the bubble motion could be chaotic and the incoherent effects would never vanish.

8. Summary

In this paper a Vlasov equation is developed to model pressure wave propagation in bubbly fluids. The advantage of this formulation is that it is fully nonlinear and can handle both coherent and incoherent bubble oscillations. The incoherent bubble oscillations may arise because of nonlinear effects, initial conditions, or because the bubbly fluid has a size distribution of bubbles. We show that if we linearize our model we can recover the dispersion relation derived by Carstensen & Foldy (1947). In addition, if we restrict our Vlasov equations to bubble oscillations which are strictly coherent then we recover the effective equations derived by van Wijngaarden (1968) and Caflisch *et al.* (1985*a*). We consider the initial value problem for an ideal bubbly flow and show that pressure waves will decay in time whereas the bubbles will continue to oscillate; the bubble oscillations become incoherent, however. The incoherence is due to the fact that the bubbles of different sizes oscillate at different frequencies. It is shown that nonlinear effects can also make a contribution to the bubble oscillations; they will become incoherent as bubble oscillations of different amplitudes will oscillate at different frequencies. We compute the damping rate for an ideal bubbly flow with identically sized bubbles. We also modify our Vlasov equation to model effects of liquid viscosity and heat transfer. It is found that pressure waves will decay due to single-bubble damping mechanisms as well as the effects of a bubble size distribution whereas the bubbles will ultimately only damp from single bubble damping mechanisms. Thus we see that the pressure waves can damp faster than the bubble oscillations.

I thank B. Brown for reading over the manuscript. This work was supported in part by the National Science Foundation through a Career Award.

Appendix A

Here we prove that $D(z)$ given by (5.26) has no zeros. This result is obtained by recognizing $D(z)$ to be an upper-analytic function (D is analytic for $\mathrm{Im}(z) > 0$). We then show that there are no zeros of $D(z)$ using the argument principle.

Since

$$D(z) = k^2 + 3\chi \quad \text{as} \quad |z| \to \infty, \tag{A 1}$$

it follows that the image of the real axis under $D(z)$ is a closed curve. The total number of zeros of $D(z)$ in the upper-half-plane is given by the number of times this curve wraps around the origin. Let $\mathscr{C}$ henceforth denote the real axis of the complex z-plane; the winding number of the image of $\mathscr{C}$ under D is then

$$M = \lim_{v\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{D'(u+\mathrm{i}v)}{D(u+\mathrm{i}v)}\,\mathrm{d}u.$$

The image of the real axis under D is given by

$$\mathscr{C}_D = D(\mathscr{C}) = \{D_R(u) + \mathrm{i}D_I(u) | -\infty < u < \infty\}.$$

$\mathscr{C}_D$ is a closed curve in the plane (D_R, D_I).

From our assumptions on $g(\beta)$, it follows from (5.34) that

$$\lim_{u\to\pm\infty} D_I(u) = 0 \quad \text{and} \quad D_I(0) = 0 \tag{A 2}$$

It also follows that $D_I(u)$ can only change sign at $u = 0$. This implies that the curve $\mathscr{C}_D$ only crosses the line $D_I = 0$ at $D_R = 1$ and $D_R = D_R(0)$. It is clear then if $D_R(0) < 0$ that $M = 1$ and $D(z)$ has one zero with $\mathrm{Im}(z) > 0$. Since $\bar{D}(z) = D(\bar{z})$, this zero must be purely imaginary. On the other hand, if $D_R(0) > 0$ then $\mathscr{C}_D$ never winds around the origin. From the expression for $D_R(u)$ (see (5.26)) we find that $D_R(0) = k^2 > 0$; consequently $D(z)$ has no zeros in the upper-half-plane. Since $\bar{D}(z) = D(\bar{z})$ there are no zeros in the lower-half-plane.

Appendix B

In this appendix we shall deduce a kinetic description in terms of the canonical variables. We first return the finite-N case where all the bubbles have the same equilibrium size ($\beta_k = 1$, for all k). The generalized momentum is then

$$z_k = \frac{\partial L}{\partial \dot{r}_k} = r_k^3 \dot{r}_k - \frac{3\kappa}{N} \sum_j G(\boldsymbol{x}_j - \boldsymbol{x}_k) r_j^2 \dot{r}_j. \tag{B 1}$$

The above equation is linear in z_k and $\dot{r}_k$ and we can, in principle, solve for $\dot{r}_k$ in terms of z_k. We have

$$\dot{r}_k = \sum_j B_{jk} z_j \tag{B 2}$$

where $B_{ij} = B_{ij}(r_1, r_2, \ldots, r_N)$. One then finds that the Hamiltonian is

$$\mathscr{H}_N = \tfrac{1}{2}\sum_{i,j}^{N} z_i B_{ij} z_j + \sum_{i}^{N} u(r_i). \tag{B 3}$$

The equations of motions are then

$$\dot{r}_k = \frac{\partial \mathscr{H}_N}{\partial z_k} = w_k \quad \text{and} \quad \dot{z}_k = -\frac{\partial \mathscr{H}_N}{\partial r_k} = F_k.$$

Following steps similar to those in § 4, we let

$$f_N(r, z, \boldsymbol{x}, t) = \frac{V^*}{N}\sum_{k=1}^{N} \delta(r - r_k(t))\delta(p - z_k(t))\delta(\boldsymbol{x} - \boldsymbol{x}_k);$$

$f_N(r, z, \boldsymbol{x}, t)$ represents the bubble density in phase space. We compute the time derivative of f_N and find that it satisfies the transport equation

$$\frac{\partial f_N}{\partial t} + w\frac{\partial f_N}{\partial r} + F\frac{\partial f_N}{\partial z} = 0, \tag{B 4}$$

where w in terms of z is determined from the equation

$$z = r^3 w - 3\chi r^2 \int f_N G(\boldsymbol{x} - \boldsymbol{y}) r^2 w \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\boldsymbol{y}. \tag{B 5}$$

We can find F with w known since

$$F = \frac{\mathrm{d}z_k}{\mathrm{d}t} = \frac{\partial L}{\partial r_k}$$

which is shown to be

$$F = \tfrac{3}{2} r^2 w^2 - \frac{\partial u}{\partial r} - 6\chi r w \int f_N G(\boldsymbol{x} - \boldsymbol{y}) r^2 w \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\boldsymbol{y}. \tag{B 6}$$

We let $N \to \infty$ and define $f = \lim_{N\to\infty} f_N$ with the same requirement as before. We then find that (B 5) becomes

$$z = r^3 w + 3\chi r^2 \ell(\boldsymbol{x}, t) \quad \text{where} \quad \ell(\boldsymbol{x}, t) = -\Delta^{-1}\int r^2 w f(r, z, \boldsymbol{x}, t)\mathrm{d}z\,\mathrm{d}r. \tag{B 7}$$

It follows from (B 7) that

$$\ell(\boldsymbol{x}, t) = Q_R^{-1}(q) \quad \text{where} \quad q(\boldsymbol{x}, t) = \int \frac{z}{r} f(r, z, \boldsymbol{x}, t)\,\mathrm{d}r\,\mathrm{d}z. \tag{B 8}$$

The operator Q_R is defined as

$$Q_R \ell = -\Delta \ell + 3\chi \eta(\boldsymbol{x}) R(\boldsymbol{x}, t)\ell, \tag{B 9}$$

where

$$R(\boldsymbol{x}, t)\eta(\boldsymbol{x}) = \int f(r, z, \boldsymbol{x}, t) r \,\mathrm{d}r\,\mathrm{d}z.$$

We then combine (B 8) and (B 7) to find an expression for w. Consequently, we find

$$w = \frac{z}{r^3} - \frac{3\chi}{r} Q_R^{-1}(q). \tag{B 10}$$

Next we use (B 7), (B 8), and (B 10) in (B 6) to find

$$F = \frac{3z^2}{2r^4} - u'(r) - 3\chi Q_R^{-1}(q)\left[\frac{z}{r^2} + \tfrac{3}{2}\chi Q_R^{-1}(q)\right]. \tag{B 11}$$

We have now expressed w and F in terms of f. It follows from (B 4) that f satisfies the transport equation

$$\frac{\partial f}{\partial t} + w\frac{\partial f}{\partial r} + F\frac{\partial f}{\partial z} = 0. \tag{B 12}$$

The important difference between (4.7) and (B 12) is that (B 12) conserves volume in phase space whereas (4.7) does not. Finally we note that

$$w = \frac{\partial H}{\partial z} \quad \text{and} \quad F = -\frac{\partial H}{\partial r}$$

where

$$H = h(r,p) + 3\chi\frac{z}{r}Q_R^{-1}q - \tfrac{9}{2}\chi^2 r(Q_R^{-1}q)^2 \tag{B 13}$$

with

$$h(r,z) = \frac{z^2}{2r^3} + \tfrac{1}{3}r^3 + \frac{r^{-3\gamma+3}}{3\gamma-3}. \tag{B 14}$$

We note that $h(r,z)$ is the Hamiltonian for a single-bubble. We remark that (B 12) has the same Hamiltonian structure as the Vlasov–Poisson equation, see, for example, Holm *et al.* (1985) and Morrison (1980, 1982).

The Hamiltonian for the single-bubble $h(r,z)$ has action-angle variables, I and θ which are related to the r,z variables. Therefore we have

$$r = \mathscr{R}(\theta,I) \quad \text{and} \quad z = \mathscr{Z}(\theta,I).$$

In these variables the Hamiltonian for the single bubble takes the form $h = h(I)$ and the equations of motion for a single-bubble become

$$\dot{I} = 0 \quad \text{and} \quad \dot{\theta} = \omega(I), \tag{B 15}$$

where $\omega(I) = h'(I)$ is the nonlinear frequency. One can verify that $\omega(I)$ is a monotonically decreasing function of I with $\omega(I) \to 0$ as $I \to \infty$.

Unfortunately, we do not have explicit expressions for $\mathscr{R}$ and $\mathscr{Z}$. We note that as $I \to 0$ the amplitude of the oscillation tends to zero and single-bubble dynamics are those of a simple harmonic oscillator. Thus we can show

$$\mathscr{R}(\theta,I) \to 1 + \sqrt{\frac{2I}{\omega_0}}\cos\theta \quad \text{as} \quad I \to 0 \tag{B 16}$$

and

$$\mathscr{Z}(\theta,I) \to \sqrt{2I\omega_0}\sin\theta \quad \text{as} \quad I \to 0, \tag{B 17}$$

where $\omega_0 = \omega(0)$.

As pointed out in Ref. 1, the stability of the desynchronized state is best determined in action-angle coordinates. In these variables the kinetic equation given by (B 12) with (B 11), (B 10) and $\eta(\boldsymbol{x}) = 1$ is

$$\frac{\partial f}{\partial t} + \Omega\frac{\partial f}{\partial \theta} + \Upsilon\frac{\partial f}{\partial I} = 0, \tag{B 18}$$

where

$$\Omega = \omega(I) - 3\chi Q_R^{-1}(q)\frac{\partial \mathscr{S}}{\partial I} + \tfrac{9}{2}\chi^2[Q_R^{-1}(q)]^2\frac{\partial \mathscr{R}}{\partial I}$$

$$\Upsilon = 3\chi Q_R^{-1}(q)\frac{\partial \mathscr{S}}{\partial \theta} - \tfrac{9}{2}\chi^2[Q_R^{-1}(q)]^2\frac{\partial \mathscr{R}}{\partial \theta}$$

$$\mathscr{S} = \mathscr{S}(\theta, I) = \frac{\mathscr{Z}(\theta, I)}{\mathscr{R}(\theta, I)},$$

$$q(\boldsymbol{x}, t) = \int \mathscr{S}(\theta, I) f(\theta, I, \boldsymbol{x}, t)\, \mathrm{d}I\, \mathrm{d}\theta,$$

and

$$R(\boldsymbol{x}, t) = \int \mathscr{R}(\theta, I) f(\theta, I, \boldsymbol{x}, t)\, \mathrm{d}I\, \mathrm{d}\theta.$$

In these variables, the desynchronized solution takes the form

$$f(\theta, I, \boldsymbol{x}, t) = f_0(I).$$

It is obvious that $f_0(I)$ is a spatially homogeneous solution to (B 18) since

$$\int \mathscr{S}(\theta, I) f_0(I)\, \mathrm{d}I\, \mathrm{d}\theta = 0.$$

Appendix C

Here we prove (6.20). We start by recalling the expression for $R_0(I)$,

$$R_0(I) = \int_0^{2\pi} \mathscr{R}(I, \theta)\, \mathrm{d}\theta.$$

We rewrite the above equation using (B 15) to give

$$R_0(I) = \omega(I) \int_0^{T(I)} \mathscr{R}(I, w(I)t)\, \mathrm{d}t,$$

where $T(I) = 2\pi/\omega(I)$ is the period. Next, it follows from the equations of motion for a single bubble that

$$\frac{\mathrm{d}\mathscr{R}}{\mathrm{d}t} = \frac{\mathscr{Z}}{\mathscr{R}^3}.$$

Therefore we can rewrite the previous equation as

$$R_0(I) = 2\omega(I) \int_{a(I)}^{b(I)} \frac{\mathscr{R}^4}{\mathscr{Z}}\, \mathrm{d}\mathscr{R},$$

where $a(I)$ and $b(I)$ are the minimum and maximum values of $\mathscr{R}$. Since $\mathscr{R}$ and $\mathscr{Z}$ are related by (B 14) we may solve for $\mathscr{Z}$ in terms $\mathscr{R}$ using (B 14) to obtain

$$R_0(I) = 2\omega(I) \int_{a(I)}^{b(I)} \frac{\mathscr{R}^{5/2}\, \mathrm{d}\mathscr{R}}{\sqrt{2(h(I) - u(\mathscr{R}))}}. \tag{C 1}$$

Using similar steps we show that

$$\frac{N(I)}{\omega(I)} = \int_0^{T(I)} \frac{\mathscr{Z}^2(I, w(I)t)}{\mathscr{R}^2(I, w(I)t)}\, \mathrm{d}t = 2\int_{a(I)}^{b(I)} \mathscr{R}^{5/2}\sqrt{2(h(I) - u(\mathscr{R}))}\mathrm{d}\mathscr{R} \tag{C 2}$$

We find, by differentiating (C 2), that

$$\frac{\mathrm{d}}{\mathrm{d}I}\left(\frac{N(I)}{\omega(I)}\right) = 2h'(I) \int_{a(I)}^{b(I)} \frac{\mathscr{R}^{5/2}\, \mathrm{d}\mathscr{R}}{\sqrt{2(h(I) - u(\mathscr{R}))}}. \tag{C 3}$$

The boundary terms that arise when differentiating (C 2) vanish because $a(I)$ and $b(I)$ are the minimum and maximum values of $\mathscr{R}$; consequently $h(I) - u(a(I)) = 0$ and $h(I) - u(b(I)) = 0$. Since $h'(I) = \omega(I)$ we see (C 3) is identical to (C 1) and thus (6.20) is proven.

Appendix D

Here we prove that if $f_0(I) = f_s(\sigma^{-1}I)/2\pi\sigma$ then (6.22) becomes

$$D_R(u) = 1 - \frac{3\chi}{k^2 + 3\chi}\left(\frac{\omega_0^2}{\omega_0^2 - u^2} + O(\sigma)\right).$$

To prove this we recall that

$$D(z) = 1 + \frac{6\pi\chi}{k^2 + 3\chi\overline{R_0}}\sum_{m=1}^{\infty} D_m(z),$$

where

$$D_m(z) = \int_0^{\infty} \frac{2m^2 s_m(I)s_{-m}(I)\omega(I)f_0'(I)\,\mathrm{d}I}{m^2\omega^2(I) - z^2}.$$

We can rewrite D_m, using (6.15), integration by parts and the substitution $I = \sigma\alpha$, as

$$D_m(z) = -\frac{1}{2\pi}\int_0^{\infty} f_s(\alpha)G_m(\sigma\alpha, z)\,\mathrm{d}\alpha,$$

where

$$G_m(I, z) = \frac{\mathrm{d}}{\mathrm{d}I}\left(\frac{2m^2 s_m(I)s_{-m}(I)\omega(I)}{m^2\omega^2(I) - z^2}\right).$$

We expand $G_m(\sigma\alpha, z)$ in a Taylor series for small σ to obtain

$$D_m(z) = -\frac{G_m(0, z)}{2\pi} + O(\sigma).$$

It follows from (B 16) and (B 17) combined with perturbation theory that

$$s_1(I)s_{-1}(I) = \tfrac{1}{2}\omega_0 I + O(I^2)$$

and

$$s_m(I)s_{-m}(I) = O(I^m), \quad m = 2, 3, 4, \ldots.$$

We can use these expressions to compute $G_m(0, z)$ from which we find

$$D_1(z) = -\frac{1}{2\pi}\left(\frac{\omega_0^2}{\omega_0^2 - z^2} + O(\sigma)\right),$$

$$D_m(z) = O(\sigma^{m-1}) \quad m = 2, 3, \ldots.$$

Similarly we can show that

$$\overline{R_0} = 1 + O(\sigma).$$

Collecting these results we have

$$D(z) = 1 - \frac{3\chi}{k^2 + 3\chi}\left(\frac{\omega_0^2}{\omega_0^2 - z^2} + O(\sigma)\right).$$

Next we let $z = u + \mathrm{i}v$ in the above expression and take $v \to 0^+$ and we obtain (6.26).

Appendix E

The goal of this appendix is to show that

$$\lim_{\sigma\to 0} D_I(\Omega) = 0 \tag{E 1}$$

when $f_0(I)$ is given by (6.15).

We start with the following results that can be deduced from the single-bubble Hamiltonian:

$$r_{max} \propto h^{1/3}, \quad z_{max} \propto h, \quad \text{and} \quad s_{max} \propto h^{2/3} \quad \text{as} \quad h \to \infty, \tag{E 2}$$

where h is the energy and the subscript denotes the maximum value obtained during one period of motion. Since $I = \int z \,\mathrm{d}r$ then it follows that $I \propto h^{4/3}$; hence

$$\omega(I) \propto I^{-1/4} \quad \text{as} \quad I \to \infty. \tag{E 3}$$

Equation (E 2) also implies that

$$s_m(I) \propto I^{1/2} \quad \text{as} \quad I \to \infty. \tag{E 4}$$

We find using the assumptions on $f_0(I)$ given in Ref. 1 combined with (E 4) and (E 3) imply that $f_0(I)$ must decay to zero at least as fast as

$$I^{-K} \quad \text{as} \quad I \to \infty \quad \text{where} \quad K > \tfrac{3}{2}.$$

Henceforth we shall assume that

$$f_0(I) \propto I^{-K} \quad \text{as} \quad I \to \infty \quad \text{where} \quad K > \tfrac{3}{2}. \tag{E 5}$$

If $f_0(I)$ decays faster than I^{-K} as, for example, a Gaussian ($\exp(-I^2)$) then our estimate for $D_I(\Omega)$ will be even smaller.

Next, we shall consider $D_I(x)$ for $|x| \ll 1$ and $|x| = O(1)$. First take $|x| = O(1)$; it then follows from (6.25) that $I_c = O(1)$. This fact combined with (6.15) shows that

$$D_I(x) \propto \frac{1}{\sigma^2} f_s'(I_c/\sigma) \quad \text{as} \quad \sigma \to 0.$$

This combined with (E 5) shows

$$D_I(x) \propto \sigma^{K-1} \quad \text{for} \quad |x| = O(1) \quad \text{as} \quad \sigma \to 0. \tag{E 6}$$

Next we consider the case when $|x| \ll 1$. Equations (6.25) and (E 3) imply

$$I_c \propto |x|^{-4} \quad \text{as} \quad x \to 0.$$

It follows using (E 3), (E 4), and (6.15) in (6.24) that

$$D_I(x) \propto \frac{f_s'(\sigma^{-1}|x|^{-4})}{\sigma^2 |x|^9} \quad \text{as} \quad \sigma \to 0.$$

Since $\sigma \ll 1$ and $|x| \ll 1$ then we can use (E 5) to obtain

$$D_I(x) \propto \sigma^{K-1} |x|^{K-1/2} \quad \text{as} \quad \sigma \to 0 \quad |x| \to 0. \tag{E 7}$$

Next can use (E 6) and (E 7) together with (6.27) to establish (E 1).

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